Math 535, lecture 2, 11/1/2023

Last time: introduction, matrix groups, the matrix exponential & lie algebra.

Today: Topological groups & representations

working toward: Peter-Weyl thm.

Def: A topological group is a group G equipped with a Hausdorff topology s.t. the map $(x,y) \mapsto xy^{-1}$ is cts $G \times G \to G$.

A continuous action of a topological group on a space X is a group action $\cdot : G \times X \to X$ which is cts for the pdt topology.

Examples: (0) $\{1\}$, (1) Any group + discrete topology

(2) $(\mathbb{R}, +)$, $(\mathbb{R}^\times, \cdot)$; (3) $GL_n(\mathbb{R}) \subset \mathbb{R}^{n^2}$;

(4) $\mathbb{Q}_p$; (5) C_2^X, ... (6) $SL_n(\mathbb{Z})$ + profinite topology

Ex: If $\{G_i\}_{i\in I}$ top. grps so is $\prod_{i\in I} G_i$ in prod top.

Lemma: Suffices to assume T_1, i.e. $\{e\}\subset G$ is closed

Pf: Left action of G on itself is a cts action so topology is G-inv't; $\{e\}$ closed iff all pts are closed.

By invariance enough to separate $e \neq g$.

Just saw: $\{g\}$ is closed so $G\setminus\{g\}$ is open.

$\Rightarrow$ inverse image by xy^{-1}: $\{(x,y) \mid xy^{-1}\neq g\}$ is open in $G\times G$. Contains $\{(e,e)\}$

$\Rightarrow$ there are open sets $\overset{e\in}{U}, \overset{e\in}{V} \subset G$ s.t. $U\times V$ [⊂ this set].

set $W = U\cap V$.

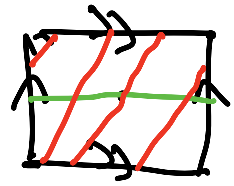
claim: $e\cdot W \cap gW = \emptyset$

pf: if $x = gy$ for $x, y\in W$ then $g = xy^{-1}$, $(x,y)\in W\times W\subset U\times V$ $\Rightarrow\!\!\!\Leftarrow$

∎

Lemma: Let $H\subset G$ be a subgroup. Then G/H is Hausdorff (wrt quotient top.) iff H is closed; $\bar{H}$ is a subgroup.

Example: $T^2 = \mathbb{R}^2/\mathbb{Z}^2$



$\mathbb{R}/\mathbb{Z} \times \{0\} \subset (\mathbb{R}/\mathbb{Z})^2$
is a closed subgp
image of line $y = mx$, m irrational is a dense subgp not closed

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Rep'n Theory

A linear representation of a group G is an action on a vector space V by linear maps (i.e. a gp hom $G \to GL(V)$)

Def: A representation π of a top. gp. G on the TVS V_π is a continuous action by linear maps

A unitary representation is one where V_π is a Hilbert space, and $\pi(g) \in \mathcal{U}(V_\pi)$ is unitary.

Def: An intertwining operator of the rep'ns (π, V), (σ, W) of G is a continuous linear operator $T: V \to W$ s.t.

$\forall g \in G: \ \sigma(g) \circ T = T \circ \pi(g).$

Write: $\mathrm{Hom}(G,H) = \{$ cts gp homs $\}$

$\mathrm{Hom}_G(\pi,\sigma) = \{$ cts G-homs $\}$

"intertwining operators"

$\rightsquigarrow$ (*) Category of top gps
(*) Category $\mathrm{Rep}(G)$ of rep'ns of G.

Example: The "standard" representations of matrix groups: $GL_n(\mathbb{R}), O(n) \curvearrowright \mathbb{R}^n$; $U(n) \curvearrowright \mathbb{C}^n$

Examples: let $O(n)$ act on S^{n-1}.
$\Rightarrow$ $O(n)$ acts on many vector spaces of functions on S^{n-1}!

Eg: $C(S^{n-1})$ (if $g \in O(n)$ close to 1, $f \in C(S^{n-1})$ " " 0 then $x \mapsto f(g^{-1}x)$ is close to 0.)

similarly actions on $L^p(S^{n-1})$, esp. $L^2(S^{n-1})$ which is unitary.

Also $C^\infty(S^{n-1})$, Sobolev spaces ~

Also image of $\mathbb{R}[x_1,..,x_n]$ in these spaces (contained in all examples)

① Observe: $\mathbb{R}[x_1,..,x_n]^{=d}$ (poly of deg d) is $O(n)$-inv't + f.d

⇒ have a dense sum of inv't subspaces

$$\bigoplus_d \mathbb{R}[x_1,..,x_n]^{=d}$$