Math 535: Real Groups

① Introduction 9/1/2023

Examples: closed matrix groups =
closed subgroups of $GL_n(\mathbb{R})$ ("general linear group")
$= \{g \in M_n(\mathbb{R}) \mid \det(g) \neq 0\}$

Goals: (1) Structure
(2) (Group actions)
(3) Representations

Most of the course: compact Lie groups

Plan: (1) topological groups, abstract rep'n theory of compact groups

(2) differential geometry & Lie groups

(3) compact Lie groups: structure theory & rep'n theory

(4) semisimple Lie groups, symmetric spaces
∞-dim rep'n theory.

Examples: $SL_n(\mathbb{R}) = \{ g \in M_n(\mathbb{R}) \mid \det g = 1 \}$
("special linear group")

$O(n) = \{ g \in M_n(\mathbb{R}) \mid {}^t g \cdot g = Id \}$
("orthogonal group") here $\det(g) \in \{\pm 1\}$

$\supseteq SO(n) = O(n) \cap SL_n(\mathbb{R})$
("special orthogonal group")

$U(n) = \{ g \in M_n(\mathbb{C}) \mid g^\dagger g = Id \} \supset SU(n)$
("unitary group") $\quad (g^\dagger = {}^t\bar{g})$

Facts: $O(n) \subset GL_n(\mathbb{R})$, $\quad U(n) \subset GL_n(\mathbb{C})$
$SO(n) \subset SL_n(\mathbb{R})$ $\quad SU(n) \subset SL_n(\mathbb{C})$
are maximal compact subgroups.

Tool: $\exp: M_n(\mathbb{R}) \to GL_n(\mathbb{R})$

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

local inverse: $\log(I+Y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} Y^n$

$\Rightarrow$ local diffeo

treat $M_n(\mathbb{R})$ as "logarithm" of $GL_n(\mathbb{R})$

① If X, Y commute, $\exp(X+Y) = \exp(X)\exp(Y)$

② to first order $\exp(tX)\exp(tY) = I + t(X+Y)$
to 2^{nd} order:

$$\exp(tX)\exp(tY) \approx \left(1 + tX + \frac{t^2}{2}X^2\right)\left(1 + tY + \frac{t^2}{2}Y^2\right)$$

$$\approx 1 + t(X+Y) + \frac{t^2}{2}\left(X^2 + Y^2 + 2XY\right)$$

$$= 1 + t(X+Y) + \frac{t^2}{2}\left(X^2 + Y^2 + XY + YX\right)$$

$$+ \frac{t^2}{2}(XY - YX)$$

$$\Rightarrow \exp(tX)\exp(tY) \approx \exp(t(X+Y)) \cdot \exp\left(\frac{t^2}{2}[X,Y]\right)$$

where $[X,Y] = XY - YX$ ("commutator")

Ex: $(M_n(\mathbb{R}), [\cdot,\cdot])$ is a Lie algebra

If $G \subset M_n(\mathbb{R})$ closed subgroup,
via log get a subspace $\mathfrak{g} = \mathrm{Lie}\,G \subset M_n(\mathbb{R})$
diffeo to nbd of 1 in G.

eg: $\det(I + tX) = 1$ to 1^{st} order
$\Leftrightarrow \mathrm{tr}\, X = 0$

$(I + s\,{}^{t}X)(I + sX) \approx Id \Leftrightarrow {}^{t}X + X = 0$

⋮

Examples: $G = U(1) \supset SO(2) \cong S^1$
$\mathrm{Lie}\, G \cong \mathbb{R}$

$\exp : \mathbb{R} \to S^1 \cong \mathbb{R}/\mathbb{Z}$
$\exp(x) = e^{2\pi i x}$

$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ (upper triangular unipotent)

$n = \left\{ \begin{pmatrix} 0 & & * \\ 0 & \ddots & \\ & & 0 \end{pmatrix} \right\}$ (" " nillpotente)

<u>Later</u> $X \to \exp(X)$ is a co-ordinate patch
$\mathfrak{g} \mapsto G$

also if $\{X_i\}_{i=1}^{d} \subset \mathfrak{g}$ basis

$\prod_{i=1}^{d} \exp(t_i X_i)$

$O(n)(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid {}^t g \cdot g = Id \}$

$\subset \mathbb{C}^{n^2}$

$\text{Lie}(U(n)) = \{ X \in M_n(\mathbb{C}) \mid {}^t\bar{X} = -X \}$