Lior Silberman's Math 535, Problem Set 4: Linear Lie groups

Closed matrix groups

Write $\mathfrak{gl}_n\mathbb{R} \overset{\text{def}}{=} M_n(\mathbb{R})$ which we will interpret as the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ with commutator bracket $[X,Y] = XY - YX$ (addition and multiplication of matrices).

1. (The matrix exponential). Fix a norm on $\mathbb{R}^n$ and a corresponding operator norm on $M_n(\mathbb{R})$.
   (a) Show that $\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!}$ converges for all $X \in M_n(\mathbb{R})$ and that $\exp(X+Y) = \exp(X)\exp(Y)$ if X,Y commute.
   (b) Conclude that $\exp((s+t)X) = \exp(sX)\exp(tX)$ for all $s,t \in \mathbb{R}$ and in particular that $\exp(X)$ is invertible for all X.
   (c) Show that $\frac{d}{dt}\exp(tX) = X\exp(tX)$.
   (d) Let $\{a_k\}_{k\geq 1} \subset \mathbb{R}$ satisfy $ka_k \to t$ and let $Y_k \in M_n(\mathbb{R})$ satisfy $Y_k \to 0$. Show that $\lim_{k\to\infty}(I + a_k(X+Y_k))^k = \exp(tX)$.
   (e) Show that $\log(g) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(g-I)$ converges for $\|g-I\| < 1$. Show that $\log(\exp(X)) = X$ and $\exp(\log(g)) = g$ hold for X,g sufficiently close to $0,I$ respectively.
   (f) Conclude that exp, log are local diffeomorphisms which are inverse to each other. Compute their differentials at $X = 0$, $g = I$.
   (g) For any direct sum decomposition $M_n(\mathbb{R}) = \oplus_{i=1}^{r} V_i$ for some linear subspaces V_i, show that $(X_i)_{i\in i} \mapsto \prod_{i=1}^{r}\exp(X_i)$ defines a local diffeomorphism $M_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ (hint: inverse function theorem).
   (h) Examining the power series show that $\log(\exp tX \exp tY) = t(X+Y) + \frac{1}{2}t^2[X,Y] + O(t^3)$.

2. Now let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a closed subgroup.
   DEF A *path* in G is a differentiable curve γ: $(-\varepsilon,\varepsilon) \to M_n(\mathbb{R})$ such that $\gamma(0) = I$. A *null sequence* is a sequence $(g_i)_{i=1}^{\infty} \subset G \setminus \{I\}$ with $\lim_{i\to\infty} g_i = I$. Set:

$$\mathfrak{g}_1 = \mathbb{R} \cdot \left\{ \lim_{i\to\infty} \frac{\log g_i}{\|\log g_i\|} \mid (g_i)_{i=1}^{\infty} \text{ is a nullsequence and the limit exists} \right\}$$

$$\mathfrak{g}_2 = \{X \in M_n(\mathbb{R}) \mid \forall t \in \mathbb{R} : \exp(tX) \in G\}$$

$$\mathfrak{g}_3 = \left\{\gamma'(0) \mid \gamma \text{ is a path in } G\right\}.$$

   (a) Show that $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3 \subset \mathfrak{g}_1$. (hint: for the first inclusion use 1(d) after raising g_i to appropriate powers).
   DEF We write $\mathfrak{g}$ for this subspace and call it the *linear* Lie *algebra* of G.
   (b) Show that $\mathfrak{g}$ is a subspace of $\mathfrak{gl}_n\mathbb{R}$ (hint: show that $\mathfrak{g}_2$ is closed under rescaling and that $\mathfrak{g}_3$ is closed under addition).
   (c) Show that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n\mathbb{R}$.
   (d) Since exp is locally invertible, to show that $\exp\restriction_{\mathfrak{g}} : \mathfrak{g} \to G$ is a local homeomorphism it's enough to show it's localy surjective. For this fix a complement $\mathfrak{h}$ such that $\mathfrak{g} \oplus \mathfrak{h} = M_n(\mathbb{R})$. Suppose that for each $r > 0$ there is g_i with $\|g_i - I\| \leq r$ and g_i is not in the image $\exp(B_1(I) \cap \mathfrak{g})$. Using 1(g) to write $g_i = \exp(X_i)\exp(Y_i)$ with $X_i \in \mathfrak{g}$ and $Y_i \in \mathfrak{h}$ both tending to zero. Show that Y_i are zero from some point onward, and obtain a contradiction.
   (c) Show that $\frac{1}{t^2}\log(\exp(tX)\exp(tY)\exp(-tX)\exp(-tY)) = [X,Y] + O(t)$ and conclude that $[X,Y] \in \mathfrak{g}$, so $\mathfrak{g}$ is a Lie subalgebra of $M_n(\mathbb{R})$.

Linear groups over fields

Fix a field F of characteristic not equal to 2, and let V be a finite-dimensional F-vectorspace.

DEFINITION. A *quadratic form* is a non-degenerate symmetric bilinear form $V \times V \to F$. A *symplectic form* is a non-degenerate antisymmetric bilinear form.

3. (Orthogonal groups over fields) Fix a quadratic form on V where $\dim V = n$.
   (a) Show that the quadratic forms on F^n are exactly those of the form $\langle u, v\rangle = \sum_{i,j} u_i Q_{ij} v_j$ where $Q \in M_n(F)$ is symmetric and of full rank.
   (b) (The polarization identity) Given a quadratic form define $q(v) = \langle v, v\rangle$. Show that we can recover $\langle u, v\rangle$ from the quadratic polynomial q (hint: consider $q(u+v)$).
   (c) Find a nondegenerate quadratic form on F^2 for which there is a vector v with $q(v) = 0$ ("isotropic vector").
   (d) Given a subspace $W \subset V$ define $W^\perp = \{v \in V \mid \forall w \in W : \langle v, w\rangle = 0\}$. Show that $\dim W + \dim W^\perp = \dim V$. The previous example shows that it's possible that $W^\perp = W$.
   (e) Show that there is a basis $\{e_i\}_{i=1}^{\dim V} \subset V$ such that $\langle e_i, e_j\rangle = 0$ if $i \neq j$ and such that $\langle e_i, e_i\rangle \neq 0$.
   DEF Write $\mathrm{O}_Q(F)$ for orthogonal group associated with a quadratic form Q:
   $$\mathrm{O}_Q(F) = \{g \in \mathrm{GL}(V) \mid \forall v \in V : q(gv) = q(v)\}$$
   (f) Show that when $F = \mathbb{R}$ up to isomorphism the only orthogonal groups are
   $$\mathrm{O}(p,q) = \mathrm{O}_{I_{p,q}}(\mathbb{R}), \qquad I_{p,q} = \begin{pmatrix} I_p & 0_{pq} \\ 0_{qp} & -I_q \end{pmatrix}.$$
   In particular we write $\mathrm{O}(n) = \mathrm{O}(n,0)$.
   FACT ("Sylvester's law of intertia") The different quadratic forms $I_{p,q}$ are not equivalent to each other. The only isomorphisms among the $\mathrm{O}(p,q)$ is that $\mathrm{O}(p,q) \simeq \mathrm{O}(q,p)$.

4. (Symplectic groups over fields) Fix a *symplectic form* on V.
   (a) (Darboux's Theorem) Show that there is a basis $\{e_i\}_{i=1}^n \cup \{f_i\}_{i=1}^n$ such that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and such that $\langle e_i, f_j\rangle = \delta_{ij}$. In particular, $\dim_F V$ is even.
   (b) The *symplectic group* is the associated symmetry group
   $$\mathrm{Sp}_{\langle\cdot,\cdot\rangle}(F) = \{g \in \mathrm{GL}(V) \mid \forall \underline{u}, \underline{v} \in V : \langle g\underline{u}, g\underline{v}\rangle = \langle \underline{u}, \underline{v}\rangle\}.$$
   Show that up to conjugacy this group does not depend on the choice of symplectic form.
   (c) Given $\underline{u} \in V$ and $a \in F$, a *symplectic transvection* is the map $U_{\underline{u},a}(\underline{v}) = \underline{v} + a\langle \underline{v}, \underline{u}\rangle \underline{u}$. Show that $U_{\underline{u},a} \in \mathrm{Sp}_{\langle\cdot,\cdot\rangle}(F)$.
   (d) Show that the representation of the symplectic group of V on V is irreducible.
   DEF Write $\mathrm{Sp}_{2n}(F)$ for the symplectic group with respect to the *standard form*: $\mathrm{Sp}_{2n}(F) = \left\{g \in \mathrm{GL}_{2n}(F) \mid g^T X g = X\right\}$ where $X = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ (I_n is the $n \times n$ identity matrix).

5. Show that the following are closed subgroups of $\mathrm{GL}_n(\mathbb{R})$ and determine their Lie algebras.
   (a) The *special linear group* $\mathrm{SL}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid \det g = 1\}$.
   (b) The *orthogonal groups*.
   (c) The *symplectic group*.

(d) The *special orthogonal groups* $\mathrm{SO}(p,q) = \mathrm{O}(p,q) \cap \mathrm{SL}_{p+q}(\mathbb{R})$.

6. Show that $\exp\colon {}_2\mathbb{R} \to \mathrm{SL}_{\neq}(\mathbb{R})$ is not surjective.