## Lior Silberman's Math 412: Problem Set 8 (due 18/3/2023)

### Practice

M1. Find the characteristic and minimal polynomial of each matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

M2. Show that $\begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are similar. Generalize to higher dimensions.

### The Jordan Canonical Form

1. For each of the following matrices, (i) find the characteristic polynomial and eigenvalues (over the complex numbers), (ii) find the eigenspaces and generalized eigenspaces, (iii) find a Jordan basis and the Jordan form.

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   RMK I suggest computing by hand first even if you later check your answers with a CAS.

2. Suppose the characteristic polynomial of $T$ is $x(x-1)^3(x-3)^4$.
   (a) What are the possible minimal polynomials?
   (b) What are the possible Jordan forms?

3. Let $T, S \in \text{End}_F(V)$.
   (a) Suppose that $T, S$ are similar. Show that $m_T(x) = m_S(x)$.
   (b) Prove or disprove: if $m_T(x) = m_S(x)$ and $p_T(x) = p_S(x)$ then $T, S$ are similar.

4. Let $F$ be algebraically closed of characteristic zero. Show that every $g \in \text{GL}_n(F)$ has a square root, in that $g = h^2$ for some $h \in \text{GL}_n(F)$.

5. Let $V$ be finite-dimensional, and let $\mathcal{A} \subset \text{End}_F(V)$ be an $F$-subalgebra, that is a subspace containing the identity map and closed under multiplication (composition). Suppose that $T \in \mathcal{A}$ is invertible in $\text{End}_F(V)$. Show that $T^{-1} \in \mathcal{A}$.

6. (The additive Jordan decomposition) Let $V$ be a finite-dimensional vector space, and let $T \in \mathrm{End}_F(V)$.
   DEF An *additive Jordan decomposition* of $T$ is an expression $T = S + N$ where $S \in \mathrm{End}_F(V)$ is diagonable, $N \in \mathrm{End}_F(V)$ is nilpotent, and $S, N$ commute.
   (a) Suppose that $F$ is algebraically closed. Separating the Jordan form into its diagonal and off-diagonal parts, show that $T$ has an additive Jordan decomposition.
   (b) Let $S, S' \in \mathrm{End}_F(V)$ be diagonable and suppose that $S, S'$ commute. Show that $S + S'$ is diagonable.
   (c) Show that a nilpotent diagonable linear transformation vanishes.
   (d) Suppose that $T$ has two additive Jordan decompositions $T = S + N = S' + N'$. Show that $S = S'$ and $N = N'$.

## Supplementary problems: $\ell^p$ spaces

A. For $\underline{v} \in \mathbb{C}^n$ and $1 \le p \le \infty$ let $\|\underline{v}\|_p$ be as defined in class.
   (a) For $1 < p < \infty$ define $1 < q < \infty$ by $\frac{1}{p} + \frac{1}{q} = 1$ (also if $p = 1$ set $q = \infty$ and if $p = \infty$ set $q = 1$). Given $x \in \mathbb{C}$ let $y(x) = \frac{\bar{x}}{|x|} |x|^{p/q}$ (set $y = 0$ if $x = 0$), and given a vector $\underline{x} \in \mathbb{C}^n$ define a vector $\underline{y}$analogously.
      (i) Show that $\left\|\underline{y}\right\|_q = \|\underline{x}\|_p^{p/q}$.
      (ii) Show that for this particular choice of $vy$, $|\sum_{i=1}^n x_i y_i| = \|\underline{x}\|_p \left\|\underline{y}\right\|_q$
   (b) Now let $\underline{u}, \underline{v} \in \mathbb{C}^n$ and let $1 \le p \le \infty$. Show that $|\sum_{i=1}^n u_i v_i| \le \|\underline{u}\|_p \|\underline{v}\|_q$ (this is called *Hölder's inequality*).
   (c) Conlude that $\|\underline{u}\|_p = \max\left\{ |\sum_{i=1}^n u_i v_i| \mid \|\underline{v}\|_q = 1 \right\}$.
   (d) Show that $\|\underline{u}\|_p$ is a seminorm (hint: A(c)) and then that it is a norm.
   (e) Show that $\lim_{p\to\infty} \|\underline{v}\|_p = \|\underline{v}\|_\infty$ (this is why the supremum norm is usually called the $L^\infty$ norm).

B. Let $X$ be a set. For $1 \le p < \infty$ set $\ell^p(X) = \{f \colon X \to \mathbb{C} \mid \sum_{x\in X} |f(x)|^p < \infty\}$, and also set $\ell^\infty(X) = \{f \colon X \to \mathbb{C} \mid f \text{ bounded}\}$.
   (a) Show that for $f \in \ell^p(X)$ and $g \in \ell^q(X)$ ($q$ as in A(a)) we have $fg \in \ell^1(X)$ and $|\sum_{x\in X} f(x)g(x)| \le \|f\|_p \|g\|_q$.
   (b) Show that $\ell^p(X)$ are subspaces of $\mathbb{C}^X$, and that $\|f\|_p = \left(\sum_{x\in X} |f(x)|^p\right)^{1/p}$ is a norm on $\ell^p(X)$
   (c) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence. Show that for each $x \in X$, $\{f_n(x)\}_{n=1}^\infty \subset \mathbb{C}$ is a Cauchy sequence.
   (d) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence and let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that $f \in \ell^p(X)$.
   (e) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence. Show that it is convergent in $\ell^p(X)$.

Hint for B(d): Suppose that $\|f\|_p = \infty$. Then there is a finite set $S \subset X$ with $\left(\sum_{x\in S} |f(x)|^p\right)^{1/p} \ge \lim_{n\to\infty} \|f_n\| + 1$.