Math 100C – WORKSHEET 11
MULTIVARIABLE OPTIMIZATION

1. Critical points; multivariable optimization

Definition. We say the point (x_0, y_0) is a *critical point* for the function $f = f(x, y)$ if f is defined in a neighbourhood of the point and

$$\begin{cases} \frac{\partial f}{\partial x}(x_0, y_0) = 0 \\ \frac{\partial f}{\partial y}(x_0, y_0) = 0 \end{cases}.$$

(1) How many critical points does $f(x, y) = x^2 - x^4 + y^2$ have?

(2) Find the critical points of $f(x, y) = x^2 - x^4 + xy + y^2$.

(3) (MATH 105 Final, 2013) Find the critical points of $f(x, y) = xye^{-2x-y}$.

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(4) **WARNING: in general checking along the axes only is not enough to determine if a point is a local minimum or maximum. For more on this look up the multivariable second derivative test in the reference book.**

(a) Let $f(x, y) = 4x^2 + 8y^2 + 7$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

(b) (MATH 105 Final, 2017) Let $f(x, y) = -4x^2 + 8y^2 - 3$. Find the critical point(s) of $f(x, y)$, and determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or neither ("saddle point").

(5) Find the critical points of $(7x + 3y + 2y^2)e^{-x-y}$.

2. Optimization

Fact. *The maximum and minimum of $f(x,y)$ in a domain (if they exist) must occur either at (1) a critical point; (2) a singular point; (3) the boundary.*

(6) Find the maximum of $(7x + 3y + 2y^2)e^{-x-y}$ for $x \geq 0$, $y \geq 0$,

(7) A company can make widgets of varying quality. The cost of making q widgets of quality t is $C = 3t^2 + \sqrt{t} \cdot q$. At price p the company can sell $q = \frac{t-p}{3}$ widgets.

(a) Write an expression for the profit function $f(q,t)$.

(b) How many widgets of what quality should the company make to maximize profits?

3. Constrained optimization

Fact (Method of Lagrange Multipliers). *Let $f(x,y)$ and $G(x,y)$ be two functions (the* objective function *and the* constraint*). Suppose that (x_0,y_0) is a local maximum or minimum of f* ***restricted to the curve*** *$G(x,y)=0$. Then there is a number λ (the "Lagrange multplier") so that the following equations are satisfied:*

$$\begin{cases} \frac{\partial f}{\partial x}(x_0,y_0) = \lambda \frac{\partial G}{\partial x}(x_0,y_0) \\ \frac{\partial f}{\partial y}(x_0,y_0) = \lambda \frac{\partial G}{\partial y}(x_0,y_0) \\ G(x_0,y_0) = 0 \end{cases}.$$

(8) (MATH 105 final, 2017) Use the method of Lagrange Multipliers to find the maximum value of the utility function $U = f(x,y) = 16x^{\frac{1}{4}}y^{\frac{3}{4}}$, subject to the constraint $G(x,y) = 50x + 100y - 500,000 = 0$, where $x \geq 0$ and $y \geq 0$.

(9) Labour-Leisure model: a person can choose to spend L hours a day not working ("leisure"), working $24 - L$ hours with way w. Suppose their fixed income is V dollars per day. Their consumption of goods is them $C = w(24 - L) + V$, equivalenly $C + wL = 24w + V$ (here C, L are variables while w, V are constants). If their utility function is $U = U(C, L)$ find a system of equations for maximum utility.

4. Combination problems

(10) Find the maximum and minimum values of $f(x, y) = -x^2 + 8y$ in the disc $R = \{x^2 + y^2 \leq 25\}$.

(11) (MATH 105 final, 2015) Find the maximum and minimum values of $f(x, y) = (x-1)^2 + (y+1)^2$ in the disc $R = \{x^2 + y^2 \leq 4\}$.