10. Multivariable calculus (24/11/2022)

Goals.

(1) 3d space: coordinates and graphs
(2) Partial derivatives

Last Time. Numerical solutions to ODE:

The Euler Scheme

Idea: linear approximation

To solve (approximately) $y' = f(x,y)$ on $[a,b]$, $y(a) = y_0$

(1) choose step size $h = \frac{b-a}{n}$, define steps $x_i = a + i \cdot h$

(2) iteratively set $\boxed{y_{i+1} = y_i + f(x_i, y_i)h.}$ (check: $x_n = a + nh = b$)

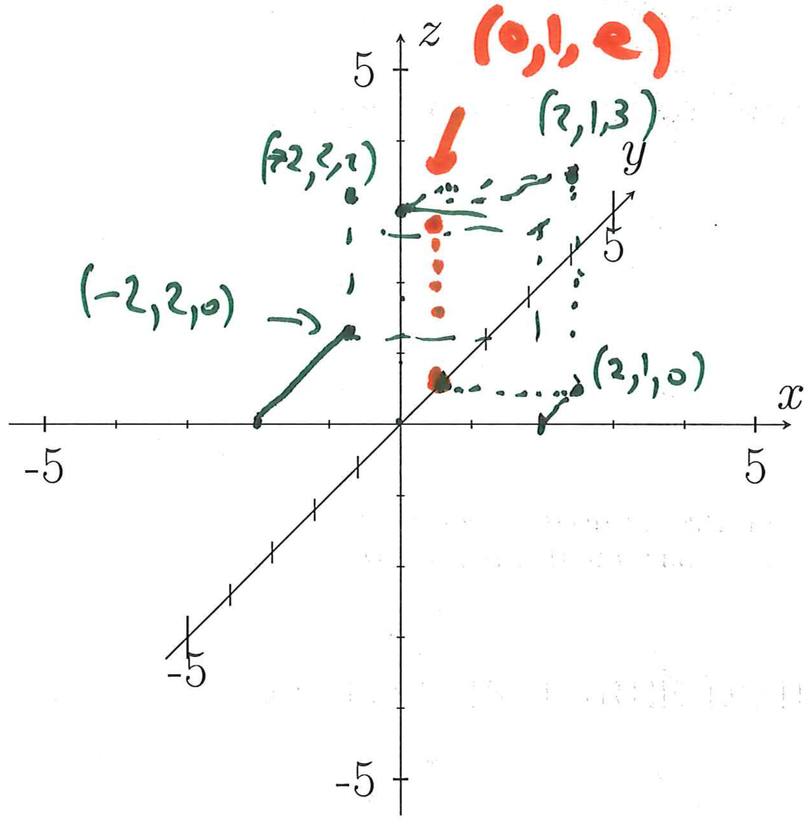
for $i = 0, 1, \ldots, n-1$.

Reference: OIL ch. 1,2.

If we have $f(x,y)$ a function of two variables, the graph $z = f(x,y)$ will "live" in 3-d space!

Math 100C – WORKSHEET 10
MULTIVARIABLE CALCULUS

1. Plotting in three dimensions



(1) Plot the points $(2, 1, 3)$, $(-2, 2, 2)$ on the axes provided.

(2) Let $f(x, y) = e^{x^2+y^2}$.

(a) What are $f(0, -1)$? $f(1, 2)$? Plot the point $(0, 1, f(0, 1))$ on the axes provided.

Date: 24/11/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$f(0,-1) = \exp(0^2 + (-1)^2) = e = f(0,1)$

$f(1,2) = \exp(1^2 + 2^2) = e^5$

(b) What is the *domain* of f (that is: for what (x, y) values does f make sense?

$e^{x^2+y^2}$ makes sense for all (x,y), so the domain is the whole plane

(c) What is the *range* of f (that is: what values does it take)?

Get all of $[1, \infty)$ (values of e^t on $[0,\infty)$)

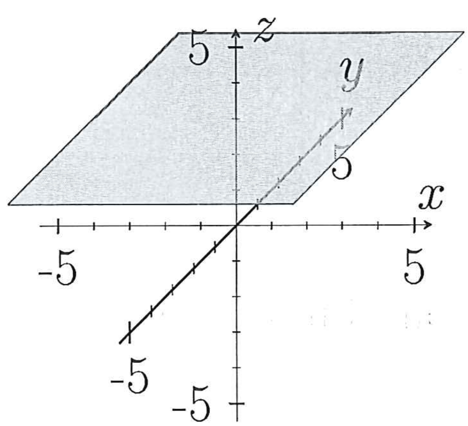
$e^0 = 1$

(3) What would the graph of $z = \sqrt{1-(x^2+y^2)}$ look like?

$\iff z^2 = 1-(x^2+y^2) \iff x^2+y^2+z^2=1$

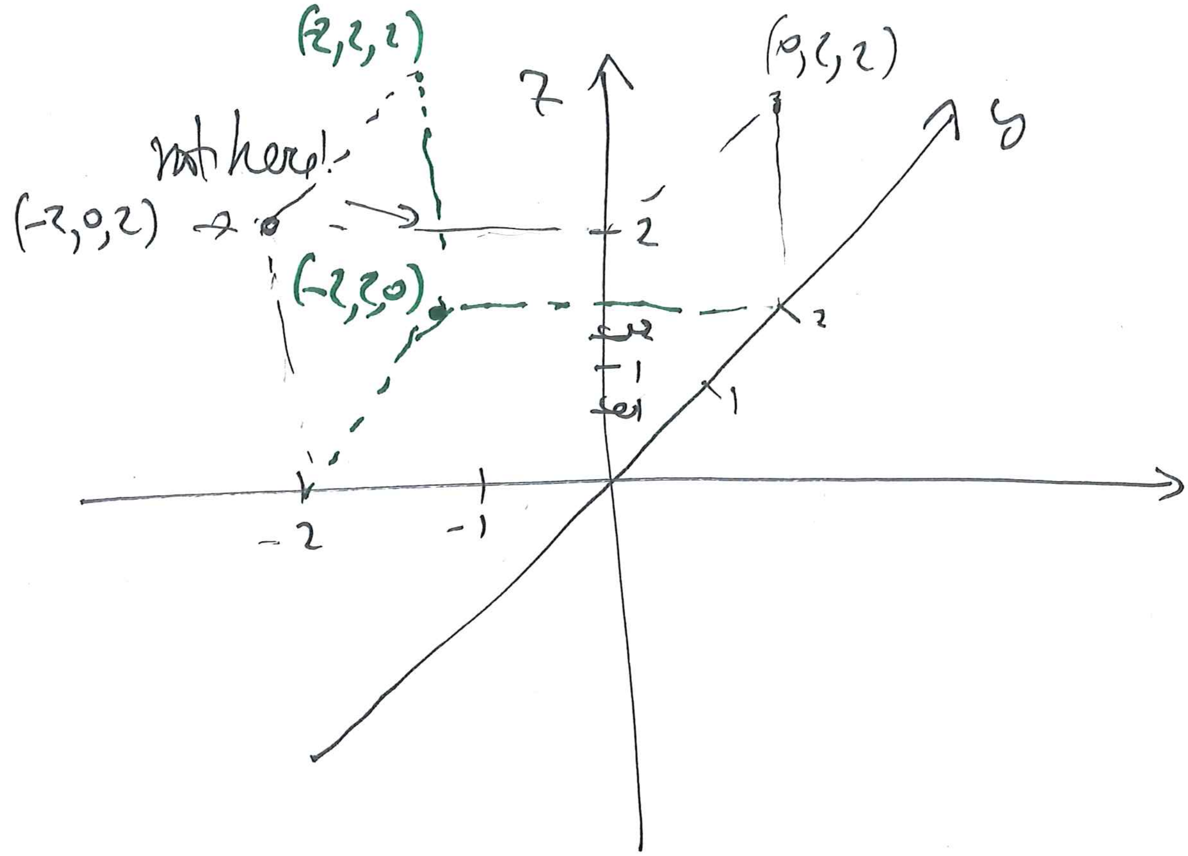
so we get the top half of the unit sphere ($z \geq 0$ here)

(4) Which plane is this?



(A) $x = 3$
(B) $y = 3$
(C) $z = 3$
(D) none
(E) not sure

Question: why isn't $(-2, 2, 2)$ at height 2 over x axis:



recall: $\sqrt{5+3} \neq \sqrt{5} + \sqrt{3}$

2. Partial derivatives

(5)(a) Let $f(x) = 2x^2 - a^2 - 2$. What is $\frac{df}{dx}$?

clearly $\frac{df}{dx} = 4x$ (a^2+2 is constant)

(b) Let $f(x) = 2x^2 - y^2 - 2$ where y is a constant. What is $\frac{df}{dx}$?

still $4x$.

(c) Let $f(x, y) = 2x^2 - y^2 - 2$. What is the rate of change of f as a function of x *if we keep y constant*?

still $4x$

notation: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = f_x = 4x$

name: "partial derivative of f w.r.t. x".

(d) What is $\frac{\partial f}{\partial y}$?

$\frac{\partial f}{\partial y} = 0 - 2y - 0 = -2y$

On notation

Sometimes have $f(x,y)$ where x, y independent

$\Rightarrow$ diff $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

Sometimes think of $f(x,y)$ where y is a function of x ("differentiation along curve"/"implicit diff")

compute $\frac{df}{dx}$ (with $y = y(x)$, using chain rule)

(Aside: $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$)

(7) One model in labour economics has a production function $Q = \left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{1/\delta}$. Here $\alpha, \delta > 0$ are parameters ($\alpha < 1$), K is the capital and E is the labour.

power law + chain rule

(a) Find the marginal product of capital: $\frac{\partial Q}{\partial K} =$

$$\frac{\partial Q}{\partial K} = \frac{1}{\delta}\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1} \frac{\partial}{\partial K}\left[\alpha K^\delta + \underbrace{(1-\alpha)E^\delta}_{\text{Constant when E is}}\right]$$

sum + power law

$$= \cancel{\frac{1}{\delta}}\left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1} \cdot \left[\alpha \cancel{\delta} K^{\delta-1}\right]$$

$$= \left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1} \cdot \alpha K^{\delta-1}.$$

(b) Find the marginal product of labour: $\frac{\partial Q}{\partial E} =$

$$\frac{\partial Q}{\partial E} = \left[\alpha K^\delta + (1-\alpha)E^\delta\right]^{\frac{1}{\delta}-1} \cdot (1-\alpha)E^{\delta-1}$$

swap K, E
α, $1-\alpha$

(6) Find the partial derivatives with respect to both x, y of

(a) $g(x, y) = 3y^2 \sin(x + 3)$

$$\frac{\partial g}{\partial x} = \underbrace{3y^2}_{\text{constant if } y \text{ is}} \frac{\partial}{\partial x} \sin(x+3) = 3y^2 \cos(x+3)$$

$$\frac{\partial g}{\partial y} = \underbrace{3\sin(x+3)}_{\text{constant if } x \text{ is}} \cdot \frac{\partial y^2}{\partial y} = 6y \sin(x+3)$$

(b) $h(x, y) = ye^{Axy} + B$

chain rule

$$\frac{\partial h}{\partial x} = y \frac{\partial}{\partial x} e^{Axy} = y \cdot e^{Axy} \cdot Ay = Ay^2 e^{Axy}$$

B is constant
here y is constant too

$$\frac{\partial h}{\partial y} = e^{Axy} + y \frac{\partial}{\partial y} e^{Axy} = e^{Axy} + ye^{Axy} \cdot Ax =$$

$$= (1 + Axy) e^{Axy}.$$

(8) We can also compute second derivatives. For example $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x} f$. Evaluate:

(a) $h_{xx} = \frac{\partial^2 h}{\partial x^2} =$

(b) $h_{xy} = \frac{\partial^2 h}{\partial y \partial x} =$ $\frac{\partial}{\partial y}\left(Ay^2 e^{Axy}\right) = 2Ay e^{Axy}$

$+ A^2 x y^2 e^{Axy}$

$= A(2Ay + A^2 xy^2) e^{Axy}$

(c) $h_{yx} = \frac{\partial^2 h}{\partial x \partial y} =$

(d) $h_{yy} = \frac{\partial^2 h}{\partial y^2} =$ $\frac{\partial}{\partial y}\left((1 + Axy) e^{Axy}\right) =$

(9) You stand in the middle of a north-south street (say Health Sciences Mall). Let the x axis run along the street (say oriented toward the south), and let the y axis run across the street. Let $z = z(x, y)$ denote the height of the street surface above sea level.

(a) What does $\frac{\partial z}{\partial y} = 0$ say about the street?

The street is level

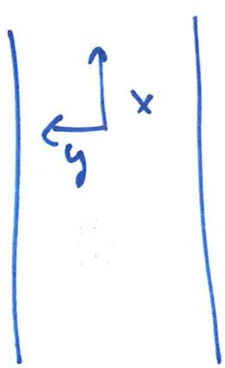
(b) What does $\frac{\partial z}{\partial x} = 0.15$ say about the street?

street is sloped with a grade of 15%.

(c) You want to follow the street downhill. Which way should you go?

south

go north



north

WW