7. Optimization (27/10/2022)

Goals.

(1) Review: calculus and the shape of the graph
(2) Optimization of functions
(3) Problem solving: optimization problems

Last Time.

(1) Taylor expansion: Approximate $f(x)$ near $x=a$ with the polynomial $T_n(x) = c_0 + c_1(x-a) + \dots + c_n(x-a)^n$.

where $\boxed{c_k = \frac{f^{(k)}(a)}{k!}}$ $\left[\Leftrightarrow T_n^{(k)}(a) = f^{(k)}(a)\right]$

Extends idea of linear approx $f(x) \approx c_0 + c_1(x-a)$

Left out: manipulating expansions

Also memorize: near 0, $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4 + \dots$

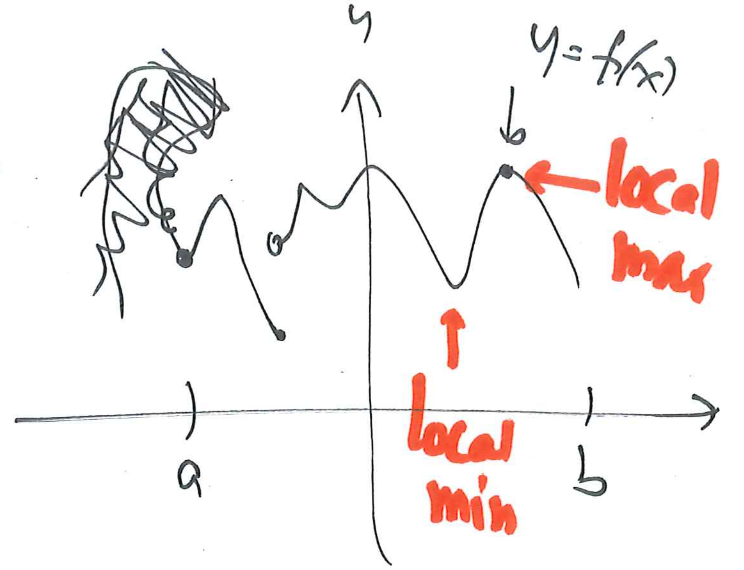
notation: $f^{(0)}(x) = f(x)$, $f^{(1)}(x) = \frac{df}{dx}$, $f^{(2)}(x) = \frac{d\left(\frac{df}{dx}\right)}{dx} = \frac{d^2f}{dx^2}$.

(2) Shape of the graph & curve sketching.

idea: by examining f (intercepts, asymptotics)
f' (>0, <0, $=0$), f'' (same)
get info on curve $y = f(x)$

$\sum_{k=0}^{n}$ means "sum for all k between 0 and n".

Optimization



Let f be defined on $[a,b]$

Call a point $x_0 \in (a,b)$

a local maximum if $f(x) \leq f(x_0)$
when x is close to x_0

a local minimum ~

(endpoints can't be local extrema)

<u>Observe:</u> if $f'(x) > 0$ f increasing near x, so x <u>not</u> local extremum

if $f'(x) < 0$ f decreasing " ", " " " " "

Conclusion: at local extremum either:

(1) $f'(x_0) = 0$ ("critical point")

(2) $f'(x_0)$ DNE ("singular point")

Call $x_0 \in [a,b]$ a global max/min if $f(x_0)$ is largest (smallest) value of f compared to <u>all</u> $f(x)$, $x \in [a,b]$

if global max is in interior (a,b) then it's a local max too.

$\Rightarrow$ If f is defined on $[a,b]$, its global extrema occur either at <u>critical</u>, <u>singular</u>, <u>or</u> <u>end</u> points.

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

Here $f'(x) = 4x^3 - 8x = 4x(x^2-2)$

So we have critical points at $0, \pm\sqrt{2}$.

Evaluating, $f(0) = 4$, $f(\pm\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 4 = 0$,

So on $[-5,5]$ the global max is 529, attained at ± 5

min is 0, attained at $\pm\sqrt{2}$

$f(\pm 5) = 529$

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

common error

Same derivative. Critical points at 0 (not at $\pm\sqrt{2}$)

$\pm\sqrt{2} \notin [-1,1]$

endpoints ± 1, $f(\pm 1) = 1$, $f(0) = 4$

global max is 4, attained at $x=0$

min 1 " $x = \pm 1$

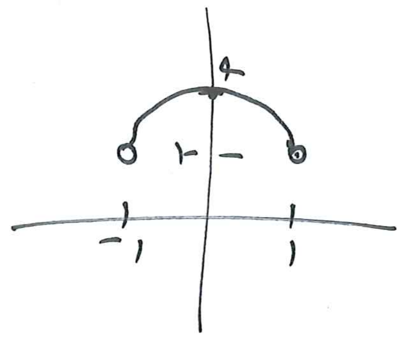
Date: 27/10/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Observe: domain affects the max/min

(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.

Now $f(x) > 1$ for all x (in $(-1,1)$)

no global min: f approaches 1, never takes value

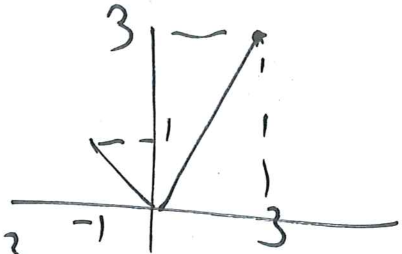


but $f(0) = 4$ still global max.

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.

(2) Let $f(x) = |x|$. Find the absolute minimum and maximum of f on the interval $[-1, 3]$.

looking at the graph:



(don't have to use Calculus)

max is 3, attained at 3
min is 0, attained at 0

But $f'(x) = \begin{cases} -1 & x<0 \\ DNE & x=0 \\ 1 & x>0 \end{cases}$ no critical pts: $f' \neq 0$ everywhere.

but $x=0$ is a singular pt.

(3) Find the global extrema (if any) of $f(x) = \frac{1}{x}$ on the intervals $(0, 5)$ and $[1, 4]$.

Optimization "in the wild"

Difficulty: going from setup to calculus and back.
(calculus not main difficulty)

If you feel "I can't get started", give quantities **names**.

Economics terminology

"Demand" = quantity that would be bought at a given price

"Revenue" = total income from sales = sum of sale prices
= quantity sold × price

"cost" = costs of production

"profit" = revenue − cost

2. Optimization problems

(4) Owners of a car rental company have determined that if they charge customers d dollars per day to rent a car, the number of cars N they rent per day can be modelled by the function $N(d) = A - Bd$ where $A, B > 0$ are constants.

(a) What is the range of d for which this model makes sense?

for $d \in [0, \frac{A}{B}]$ (Not paying people to rent cars, if $d > \frac{A}{B}$, $N(d) < 0$)

(b) What price should they set to maximize their daily *revenue*?

The revenue at price d is $R(d) = N(d) \cdot d$

$$= (A - Bd)d$$

$$= Ad - Bd^2$$

$R'(d) = A - 2Bd$, has a critical pt at $d_0 = \frac{A}{2B} \in [0, \frac{A}{B}]$

$R(0) = 0$, $R(\frac{A}{B}) = 0$, $R(\frac{A}{2B}) = \frac{A^2}{4B}$

so maximum revenue of $\frac{A^2}{2B}$ at price $\frac{A}{2B}$

(5) A car factory can produce up to 120 units per week. Find the (whole number) quantity q of units which maximizes *profit* if the total revenue in dollars is $R(q) = (750 - 3q)q$, the total cost in dollars is $C(q) = 10,000 + 148q$ (observe the combination of *fixed* and *variable* costs).

q makes sense on $[0, 120]$ (after this the price becomes negative)

On this interval, the profit is

$$P(q) = R(q) - C(q) = 750q - 3q^2 - 148q - 10,000$$

$$= 602q - 3q^2 - 10,000.$$

If P maximized at q_0, try nearest integers

(6) A ferry operator is trying to optimize profits. A ferry trip takes 1 hour and costs \$250 in fuel. The ferry can carry up to 100 cars, each paying \$50 for the trip. Worker salaries total \$500/hour. The workers can load $N(t) = 100\frac{t}{t+1}$ cars in t hours.

(a) How much time should be devoted to loading to maximize profits *per trip*.

Let t be the time devoted to loading.
In that time the workers load $N(t) = 100 \frac{t}{t+1}$ cars

(can have $0 \le t < \infty$)

The revenue is then $R(t) = 50 \cdot N(t) = 5{,}000 \frac{t}{t+1}$ dollars

Our cost is $C(t) = 250 + 500(1+t)$

So the profit is $P(t) = R(t) - C(t) =$

$= 5{,}000 \frac{t}{t+1} - 500t - 750$

as $t \to \infty$ $P(t) \sim 5{,}000 - 500t - 750 \sim -500t$

so maximum will be in the interior.

calculus..., $P'(\sqrt{10} - 1) = 0$

(b) The ferry runs continuously. How much time should be devoted to loading to maximize profits *per hour*.

Profits per hour are $\frac{P(t)}{17+t}$

$\vdots$ Calculus

maximum at $t =$