1. Expressions; Asympototics (14/9/2022)

Today's Goals.

(1) Power laws, exponentials, and their asymptotics
(2) Asymptotics of sums
(3) Parse trees
(4) Asymptotics

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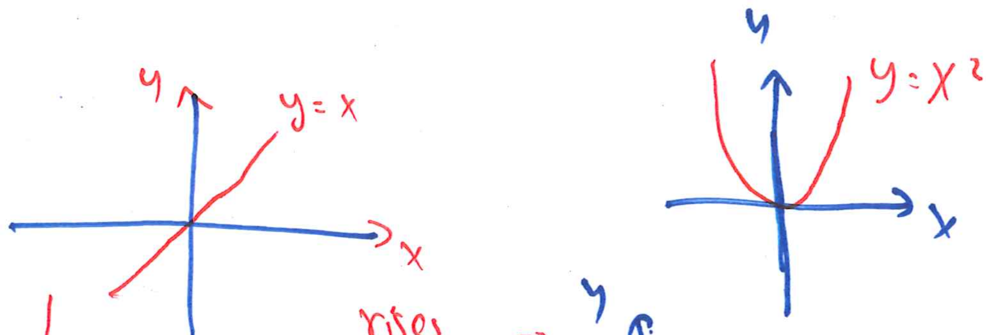
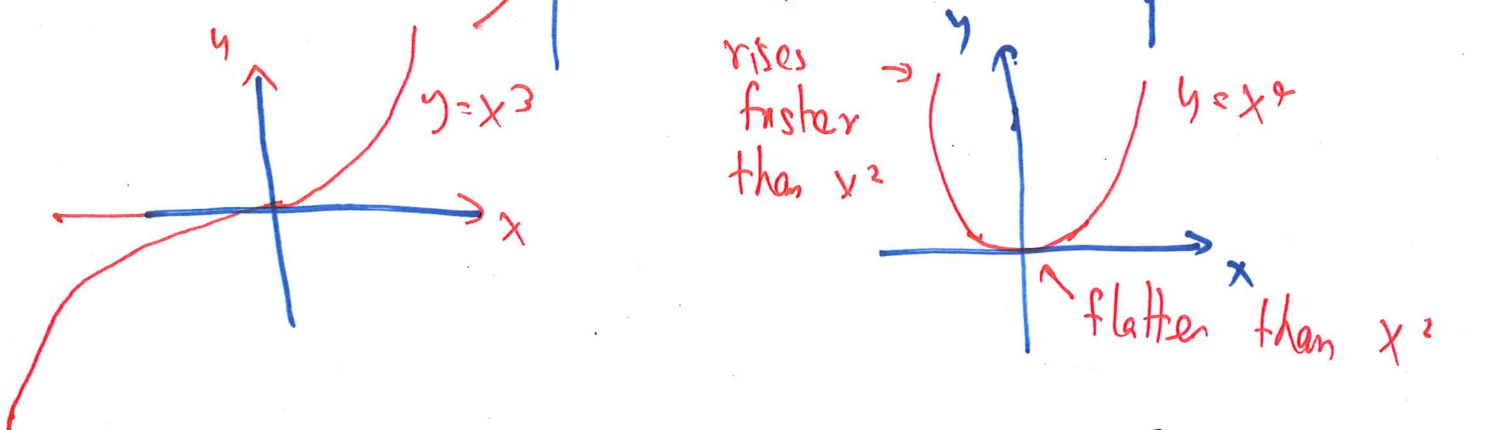
① Power laws

Expressions like x^7, x^{-5}, $x^{\frac{1}{3}}$, $18x^3$, $-\frac{1}{2}x^{-2}$

$7t^3$, ct^8, $9t^{-3}$

Graphically:





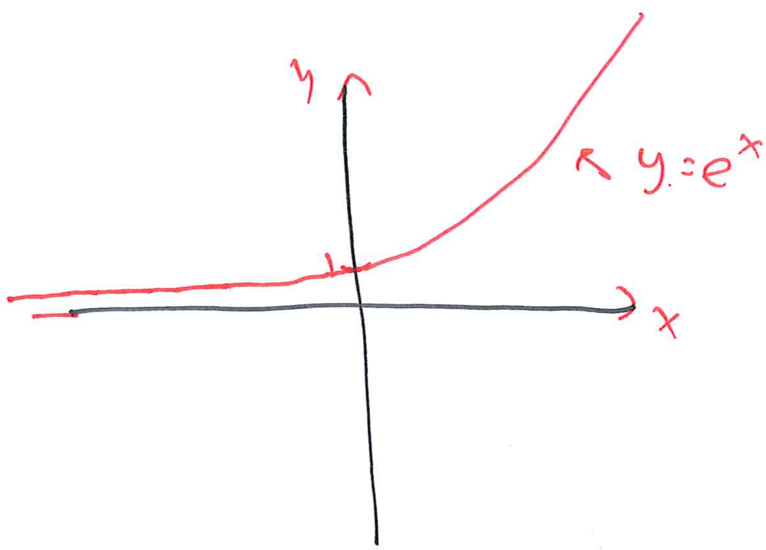
summary:
(1) different ways a quantity can change
(2) usually care when parameter is very small
or very large

② Exponentials

Expressions like b^x, 9^x, $2^{3x} = (2^3)^x$

e^{7x}, $\frac{1}{e^x}$ (why?)

$\boxed{C \cdot b^x}$



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Math 100C – WORKSHEET 1
EXPRESSIONS AND ASYMPTOTICS

1. Asymptotics: simple expressions

(1) Classify the following functions into *power laws / power functions* and *exponentials*: x^3, πx^{102}, e^{2x}, $c\sqrt{x}$, $-\frac{8}{x}$, 7^x, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x}$, $\frac{9}{x^{7/2}}$, x^e, π^x, $\frac{A}{x^b}$.

Power laws

x^3

$\pi \cdot x^{102}$

$c\sqrt{x} = c \cdot x^{1/2}$

$-\frac{8}{x} = -8 \cdot x^{-1}$

$\frac{9}{x^{7/2}} = 9 \cdot x^{-7/2}$

x^e

$\frac{A}{x^b} = A \cdot x^{-b}$

Exponentials

$e^{2x} = (e^2)^x$

7^x

$8 \cdot 2^x$

$-\frac{1}{\sqrt{3}} \cdot \left(\frac{1}{2}\right)^x$

π^x

$\boxed{\begin{array}{l}\text{but } x^x \text{ not power law} \\ \text{(variable exponent)} \\ \text{not exponential} \\ \text{(variable base)}\end{array}}$

Date: 14/9/2022, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

③ Combination of effects

Look at $x^2 + 1$.

① if x is very small (write $x \to 0$)
1 will dominate x^2 so $1 + x^2 \approx 1$
(in fact $1 + x^2 \sim 1$)

② if x is very big ($x \to \infty$)
Now x^2 dominates, $1 + x^2 \sim x^2$

Summary: if we have a sum, find the dominant piece.

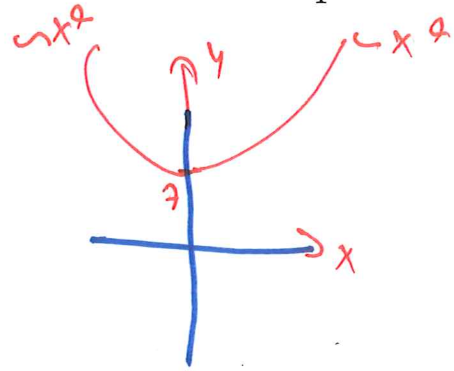
(2) How does the each expression behave when x is large? small? what is x is large but negative? Sketch a plot

(a) $7 + x^2 + x^4$

As $x \to \infty$ $\quad 7 + x^2 + x^4 \sim x^4$

As $x \to 0$ $\quad 7 + x^2 + x^4 \sim 7$

As $x \to -\infty$ $\quad 7 + x^2 + x^4 \sim x^4$



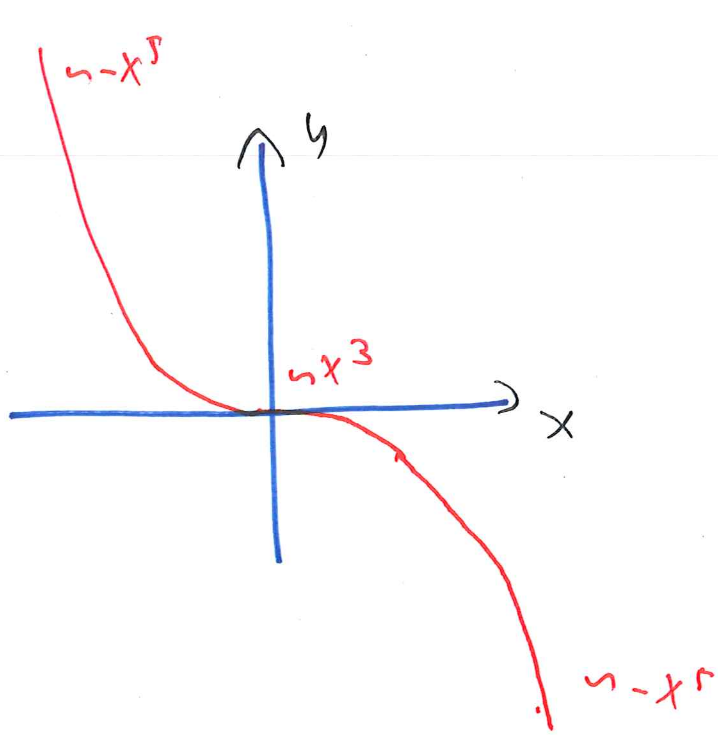
(b) $x^3 - x^5$

As $x \to \infty$ $\quad x^3 - x^5 \sim -x^5$

As $x \to 0$ $\quad x^3 - x^5 \sim x^3$

As $x \to -\infty$ $\quad x^3 - x^5 \sim -x^5$

$$\boxed{\begin{aligned} &1000 - 1{,}000{,}000{,}000 \\ &\qquad \approx -1{,}000{,}000{,}000 \end{aligned}}$$

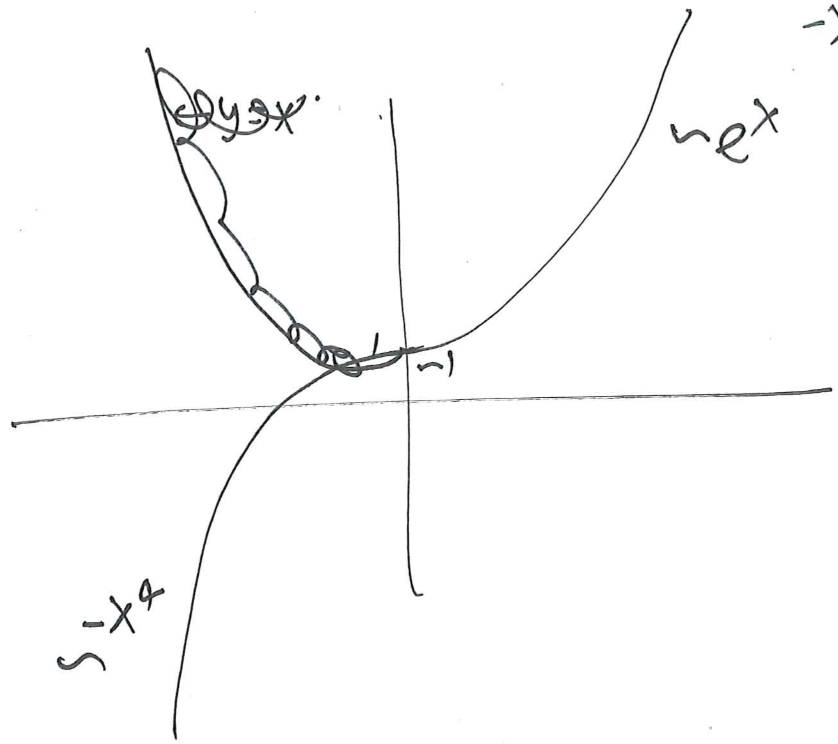


(c) $e^x - x^4$

As $x \to \infty$ $\quad e^x - x^4 \sim e^x$ $\quad$ (exponentials beat polynomials at ∞)

As $x \to 0$ $\quad e^x - x^4 \sim 1$

As $x \to -\infty$ $\quad e^x - x^4 \sim -x^4$ $\quad$ (e^x decays as $x \to -\infty$, $-x^4$ dominates it)



$$e^x = \frac{1}{e^{-x}}$$

(e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time t behaves like

$$A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \, .$$

(A, B, C are constants). Which strain dominates eventually? What would the number of infected people look like?

$1.1 > 1.05 > 0.98$

so eventually (i.e. as $t \to \infty$)

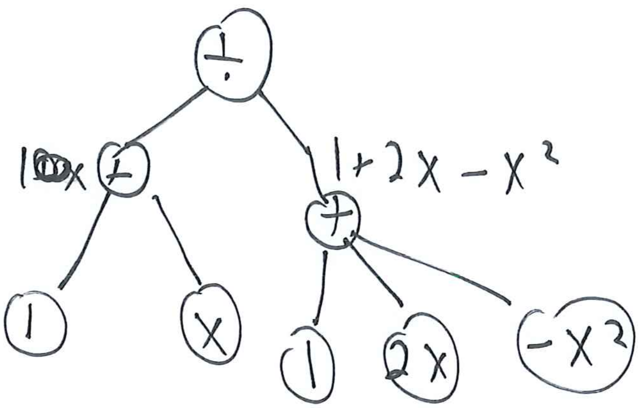
$$A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \sim B \cdot 1.1^t$$

④ Parse trees

Look at

$$\frac{1+x}{1+2x-x^2}$$

assembled from $1+x$, $1+2x-x^2$ by division



As $x \to \infty$ $\quad 1+x \sim x$

$1+2x-x^2 \sim -x^2$ $\quad$ so $\quad \dfrac{1+x}{1+2x-x^2} \sim \dfrac{x}{-x^2} \sim -\dfrac{1}{x}$

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find the way the expression is put together, go to each piece, repeat.

2. Asymptotics of complicated expresisons

(3) Construct parse trees for the following expressions:

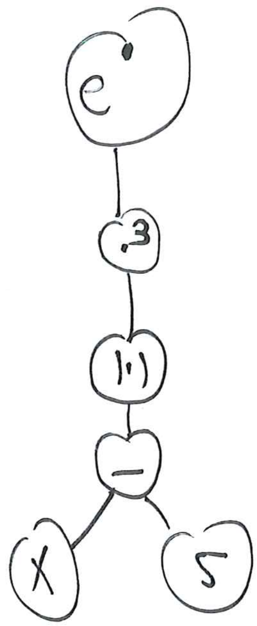
(a) $e^{|x-5|^3}$

(b) $\frac{e^x+A\sin x}{e^x-x^2}$

(c) $\frac{1+x}{1+2x-x^2}$

(d) $\left(\frac{t+\pi}{t-\pi}\right)\sin\left(\frac{t+\pi}{2}\right)$

$e^{|x-5|^3}$



for large x, $e^{|x-5|^3} \sim e^{x^3}$

if $x \to -\infty$ $e^{|x-5|^3} \sim e^{-x^3}$

if $x \to 0$, $e^{|x-5|^3} \sim e^{125}$