Lior Silberman's Math 223: Problem Set 12 (due 14/4/2021)

Practice problems

Section 6.2

M1. Check that the eigenvectors of the matrix $\begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ from PS10 are orthogonal.

For submission

1. (a) Let $\{x_i\}_{i=1}^n \subset \mathbb{R}$ be n real numbers. Applying the CS inequality to the vectors $(x_1, \ldots, x_n)$ and $(1, \ldots, 1)$, show that $\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2 \le \frac{1}{n}\sum_{i=1}^n x_i^2$.

   RMK The quantities $\frac{1}{n}\sum_{i=1}^n x_i$, $\sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2 - \left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2}$ are called respectively the *expectation* and *standard deviation* of the random variable that takes the values x_i with equal probability $\frac{1}{n}$.

   (**b) Let $\{x_i\}_{i=1}^n \subset \mathbb{R}$ be positive. The *Arithmetic Mean* of these numbers is the number $\mathrm{AM} = \frac{1}{n}\sum_{i=1}^n x_i$. The *Harmonic Mean* is the number $\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i}} = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$. Show the *inequality of the means* HM$\le$AM (with equality iff all the x_i are equal) by applying the CS inequality to suitable vectors.

2. Let $A \in M_n(\mathbb{C})$ be diagonable. Show that there exists $B \in M_n(\mathbb{C})$ such that $B^2 = A$.

3. Let $C_c^\infty(\mathbb{R})$ denote the set of functions on $\mathbb{R}$ that are infinitely differentiable and have *bounded support*: if $f \in C_c^\infty(\mathbb{R})$ then there is some interval $[-L, L]$ such that $f = 0$ outside it. Let $D\colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ be the differentiation operator. Equipe $C_c^\infty(\mathbb{R})$ with the inner product $\langle f, g\rangle = \int_{-\infty}^{+\infty} \bar{f}(x)g(x)dx$ (the integral converges since by hypothesis the functions are zero outside some finite interval). Show that $\langle f, Dg\rangle = \langle -Df, g\rangle$ (hint: this is a well-known formula).

The Quantum Harmonic Oscillator, II

Let $H = -D^2 + M_{x^2}$ act on $V = \left\{p(x)e^{-x^2/2} \mid p \in \mathbb{R}[x]\right\}$ as in PS10. Also let $V_{\mathbb{C}} = \left\{p(x)e^{-x^2/2} \mid p \in \mathbb{C}[x]\right\}$. Equip these spaces with the inner product $\langle f, g\rangle = \int_{-\infty}^{+\infty} \bar{f} g dx$.

SUPP (This problem is not for submission)

(a) Let $f, g \in V_{\mathbb{C}}$. Show that the integral $\int_{-\infty}^{+\infty} \bar{f} g dx$ converges absolutely if $f, g \in V_{\mathbb{C}}$ and defines an inner product there.

(b) Show that $\hat{p} = -iD$ is a symmetric operator on $V_{\mathbb{C}}$ in that $\langle f, \hat{p}g\rangle = \langle \hat{p}f, g\rangle$ (this notation comes from physics).

(c) Show that $\hat{x} = M_x$ is a symmetric operator on $V_{\mathbb{C}}$ in that $\langle f, \hat{x}g\rangle = \langle \hat{x}f, g\rangle$.

4*. By the supplementary problem $\langle \cdot, \cdot \rangle$ really is an inner product on $V_{\mathbb{C}}$.

(a) Show (either directly or using the results of the supplementary problem) that $\langle f, Hg\rangle = \langle Hf, g\rangle$ for all $f, g \in V_{\mathbb{C}}$.

DEF In PS10 we showed that $H(V_n) \subset V_n$ where $V_n = \left\{p(x)e^{-x^2/2} \mid p \in \mathbb{R}^{<n}[x]\right\}$. Let U_n be the orthogonal complement of V_n in V_{n+1}.

(b) Show that U_n is one-dimensional and is spanned by a function $f_n(x) = h_n(x)e^{-x^2/2}$ where $h_n \in \mathbb{R}[x]$ has degree exactly n.

(c) Use (a) to show that Hf_n is also orthogonal to V_n and conclude that f_n is an eigenfunction of H.

(d) Writing $H\left(h_n e^{-x^2/2}\right)$ in the from $pe^{-x^2/2}$ for a polynomial p, and examining the coefficient of x^n in p, show that $Hf_n = (2n+1)f_n$.

(**e) Show that the h_n are (up to normalization) exactly the Hermite polynomials of PS11.

Extra credit

P1. Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. For $\underline{u}\in V$ set $\varphi_{\underline{u}}(\underline{v})=\langle\underline{u},\underline{v}\rangle$. That $\varphi_{\underline{u}}\in V^*$ follows from the definition of the inner product.

(a) Show that the map $\Phi\colon V\to V^*$ given by $\phi(\underline{u})=\varphi_{\underline{u}}$ (warning: this is a map valued in linear maps!) is anti-linear, in that $\varphi_{c\underline{u}+\underline{u}'}=\bar{c}\varphi_{\underline{u}}+\varphi_{\underline{u}'}$.

(**b) Show that Φ is injective.
Hint: If $\underline{u}\neq\underline{0}$ show that $\varphi_{\underline{u}}$ is non-zero, and then use additivity of Φ to get injectivity from that.

(c) We proved in class that if $\dim V=n<\infty$ then Φ is surjective, hence a bijection. Show that its inverse map $V^*\to V$ is also anti-linear.

P2. (Yet another approach to the Quantum Harmonic Oscillator). Let V be a vector space equipped with operators $X,D\in\mathrm{End}(V)$ such that $[D,X]=1=\mathrm{Id}_V$ (we checked that $D=\frac{d}{dx}$ and $X=M_x$ satisfy this commutation relation in a previous problem set). Let $A=\frac{1}{\sqrt{2}}(X+D)$ ("lowering operator"), $A^\dagger=\frac{1}{\sqrt{2}}(X-D)$ ("raising operator"), and $N=A^\dagger A$ ("number operator").

WARNING X,D don't commute, so N isn't quite the same as $H=\frac{1}{2}\left(X^2-D^2\right)$, but as it turns out N isn't very different from H.

(a) Suppose V is finite-dimensional. Computing the trace $\mathrm{Tr}[X,D]$ in two different ways obtain a contradiction.

(b) Compute $\left[A,A^\dagger\right]$ and use that to show that $[N,A]=-A$ and $\left[N,A^\dagger\right]=A^\dagger$.

(c) Suppose that $Nf=\lambda f$ for some $f\in V$ and scalar λ. Show that $N(Af)=(\lambda-1)(Af)$ and $N(A^\dagger f)=(\lambda+1)(A^\dagger f)$ (that's why we call these "lowering" and "raising" operators).

(*d) Let $f_0=f$ and for $k\geq 1$ define $f_k=\left(A^\dagger\right)^k f$ and $f_{-k}=A^k f$. Show that $Nf_n=(\lambda+n)f_n$ for all n (positive or negative), and conclude that $A^\dagger f_n$ is proportional to f_{n+1} that Af_n is proportional to f_{n-1}.

(e) Conclude that the subspace $W=\mathrm{Span}\left(\{f_n\}_{n\in\mathbb{Z}}\right)$ is invariant by both $A,A^\dagger$ (they map every vector in it to another vector in it) and hence the same is true for $N=A^\dagger A$ and $H=N+\frac{1}{2}$.

P3. Continuing problem P2, suppose now that V is an inner product space and that X,D satisfy $\langle f,Xg\rangle=\langle Xf,g\rangle$ and $\langle f,Dg\rangle=-\langle Df,g\rangle$.

(a) Show that $\langle f,Ag\rangle=\left\langle A^\dagger f,g\right\rangle$ and that $\left\langle f,A^\dagger g\right\rangle=\langle Af,g\rangle$.

(b) Let f be non-zero and suppose that $Nf=\lambda f$. Show that $\lambda=\frac{\|Af\|}{\|f\|}$ and conclude that λ is a non-negative real number.

(c) Deduce from (a) and P2(d) that for n large enough $f_{-(n+1)}=0$. If $m\geq 0$ is the smallest number such that $f_{-(m+1)}=0$ show that $f_{-m}\neq 0$ but $Af_{-m}=0$. In other words W must contain a basis vector killed by A.

(c) Show that $Nf_{-m}=0$. Conclude that W must contain an eigenvector of N with eigenvalue 0 and that $\lambda-n=0$ so λ (the eigenvalue of f_0) must be a non-negative integer.

(e) Repeating the construction of P2(d),P2(e) but starting from $g_0=f_{-m}$ show that $W=\mathrm{Span}\left(\{g_k\}_{k\geq 0}\right)$ where $Ng_k=kg_k$.

(f) Show that $\langle g_{k+1},g_{k+1}\rangle=(k+1)\langle g_k,g_k\rangle$ and prove by induction that $g_k\neq 0$ for all $k\geq 1$. In other words, the eigenvalues of N are exactly the non-negative integers (if it has any). This gives further context to P2(a).

(g) Letting $h_k=\frac{1}{\|g_0\|}\frac{1}{\sqrt{k!}}g_k$ show that $\{h_k\}_{k\geq 0}$ is an orthonormal system.

(h) Let $H=\frac{1}{2}(X^2-D^2)$ (that's the operator from PS10 and problem 4 above). Show that $H=N+\frac{1}{2}$ and conclude that $Hh_k=(k+\frac{1}{2})h_k$. In particular the smallest eigenvalue of H is $\frac{1}{2}$.

RMK1 When $X = M_x$, $D = \frac{d}{dx}$ the equation $Ag_0 = 0$ becomes the differential equation $g_0' + xg_0 = 0$, and it is easy to check the solution is $e^{-x^2/2}$ up to scaling. Now applying $A^\dagger = M_x - \frac{d}{dx}$ repeatedly will produce the Hermite polynomials (multiplied by $e^{-x^2/2}$) discovered in PS10 and in problem 4 above.

RMK2 The state g_0 is called the *ground state* – the lowest energy state of the quantum harmonic oscillator. The fact that $Hg_0 = \frac{1}{2}g_0$ means that the ground state has energy $\frac{1}{2}$ rather than zero (eigenvalues of H correpsonnd to possible energies of the system). The fact that the ground state has positive energy is surprising and has non-trivial physical implications.

Supplementary problem: inequalities and induction

A. Use simple induction on n to establish *Lagrange's identity:* for all $\underline{a}, \underline{b} \in \mathbb{R}^n$:

$$\|\underline{a}\|^2 \|\underline{b}\|^2 - (\langle \underline{a}, \underline{b} \rangle)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) - \left(\sum_{i=1}^n a_i b_i\right)^2 = \sum_{1 \le i < j \le n} \left(a_i b_j - a_j b_i\right)^2$$

(note that the Cauchy–Schwarz inequality for $\mathbb{R}^n$ follows immediately)

B. (Another proof of Cauchy–Schwarz) Let $C(n)$ be the claim "the Cauchy–Schwarz inequality holds for vectors of length n".
   (a) Prove $C(2)$, for example using Lagrange's identity.
   (b) Let $\underline{x}, \underline{y}$ be vectors of length $2n$. Write $\underline{x} = (\underline{x}^1, \underline{x}^2)$ and $\underline{y} = (\underline{y}^1, \underline{y}^2)$ where the components are vectors of length n, show that $\langle \underline{x}, \underline{y} \rangle = \langle \underline{x}^1, \underline{y}^1 \rangle + \langle \underline{x}^2, \underline{y}^2 \rangle$ and that $\|\underline{x}\| = \left\| \left(\|\underline{x}^1\|, \|\underline{x}^2\| \right) \right\|$ where the outer norm is computed in $\mathbb{R}^2$.
   (c) Breaking up vectors of length $2n$ as in (b) show that $C(n)$ and $C(2)$ together imply $C(2n)$.
   (d) Prove by induction that $C(2^k)$ holds for all $k \ge 1$.
   (e) Show that $C(n)$ implies $C(n-1)$ (hint: extend $\underline{x}, \underline{y} \in \mathbb{R}^{n-1}$ to vectors of length n by making the last coordinate zero).

The proof techique of problem B is called "forward-backward induction" or "Cauchy induction".

C. For positive quantities x_i the *inequality of the means* is the statement $\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{x_i}} \le \left(\prod_{i=1}^n x_i\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^n x_i \le \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p} \le \left(\frac{1}{n}\sum_{i=1}^n x_i^q\right)^{1/r}$ (here $1 \le p \le r < \infty$) (we call the first value the "harmonic mean", the middle value the "geometric mean", the third value the "arithmetic mean" of the quantities x_i).
   (a) Applying the AM-GM inequality $\left(\prod_{i=1}^n x_i\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^n x_i$ to reciprocals $x_i = \frac{1}{y_i}$ show obtain the HM-AM inequality $\frac{1}{\frac{1}{n}\sum_{i=1}^n \frac{1}{y_i}} \le \left(\prod_{i=1}^n y_i\right)^{1/n}$. It's therefore enough to prove the AM-GM inequality.
   (b) Applying the inequality $\frac{1}{n}\sum_{i=1}^n x_i \le \left(\frac{1}{n}\sum_{i=1}^n |x_i|^{p/r}\right)^{r/p}$ to $x_i = y_i^r$ show that $p \mapsto \left(\frac{1}{n}\sum_{i=1}^n |y_i|^p\right)^{1/p}$ is an increasing function of p (the limit of this function as $p \to \infty$ was calculated in the supplement to PS11). It follows that it's enough to prove that $\frac{1}{n}\sum_{i=1}^n x_i \le \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p}$.
   (c) Let $I(n)$ be the claim $\left(\prod_{i=1}^n x_i\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^n x_i \le \left(\frac{1}{n}\sum_{i=1}^n x_i^p\right)^{1/p}$. Prove $I(2)$.
   (d) Show that $I(n)$ and $I(2)$ together imply $I(2n)$, and conclude by induction that $I(2^k)$ holds for all k.
   (e) Show that $I(n)$ implies $I(n-1)$. Note that here one has to choose the extension carefully.
   (f) Show that the inequality of the means holds for all n.

Supplementary problem: Fourier series

D. In this problem we use the standard inner product on $C(-\pi,\pi)$.

(a) Show that $\left\{\frac{1}{\sqrt{2\pi}}\right\}\cup\left\{\frac{1}{\sqrt{\pi}}\cos(nx),\frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$ is an orthonormal system there.

(b) Let a_0,a_n,b_n be the coefficient of $f(x)=2\pi|x|-x^2$ with respect to $\frac{1}{\sqrt{2n}}$, $\frac{1}{\sqrt{\pi}}\cos(nx)$, $\frac{1}{\sqrt{\pi}}\sin(nx)$. Find these.

(c) Show that for any x, the series $\frac{1}{\sqrt{2\pi}}a_0+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)$ is absolutely convergent.

FACT1 The system above is *complete*, in that the only function orthogonal to the span is the zero function. If we denote the partial sums $\left(S_Nf\right)(x)=a_0\frac{1}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\left(a_n\cos(nx)+b_n\sin(nx)\right)$, this shows $S_Nf\xrightarrow[N\to\infty]{}f$ "on average" in the sense that $\|f-S_Nf\|_{L^2(-\pi,\pi)}^2=\int_{-\pi}^{\pi}\left|f(x)-\left(S_Nf\right)(x)\right|^2\mathrm{d}x\xrightarrow[N\to\infty]{}0$ (in fact, this holds for any f such that $\int_{-\pi}^{+\pi}|f(x)|^2\mathrm{d}x<\infty$).

FACT2 For any $x\in(-\pi,\pi)$ if the sequence of real numbers $\left\{\left(S_Nf\right)(x)\right\}_{N=1}^{\infty}$ converges, and if f is continuous at x, then limit of the sequence is $f(x)$.

(d) Conclude that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, a discovery of Euler's.

Supplementary problem: The Rayleigh quotient

E. Given a matrix $A\in M_n(\mathbb{R})$ consider the function $f\colon\mathbb{R}^n\to\mathbb{R}$ given by $f(\underline{x})=\underline{x}^tA\underline{x}=\sum_{i,j=1}^{n}a_{ij}x_ix_j$. We introduce the notation $\|\underline{x}\|_2^2=\sum_{i=1}^{n}x_i^2$.

(a) Show that $\left(\nabla f\right)(\underline{x})=A\underline{x}+A^t\underline{x}$.

(b) Let $\underline{v}$ be the point where f attains its maximum on the unit sphere $S^{n-1}=\{\underline{x}\in\mathbb{R}^n\mid\|\underline{x}\|=1\}$. Use the method of Largrange multipliers to show that $\underline{v}$ satisfies $A\underline{v}+A^t\underline{v}=\lambda\underline{v}$ for some $\lambda\in\mathbb{R}$.

(c) A matrix is *symmetric* if $A=A^t$. Show that every symmetric matrix has a real eigenvalue.

(d) Show that the following two maximization problems are equivalent:

$$\max\left\{f(\underline{x})\mid\|\underline{w}\|_2=1\right\}\leftrightarrow\max\left\{\frac{f(\underline{x})}{\|\underline{x}\|_2^2}\mid\underline{x}\neq\underline{0}\right\}.$$