## Lior Silberman's Math 223: Problem Set 11 (due 4/4/2022)

### Practice problems 1: diagonalization

Section 6.1: all problems are suitable

M1. Write down some matrix $A \in M_4(\mathbb{R})$ such that $A$ has four distinct eigenvalues (your choice) with the correspoding eigenvectors being $\begin{pmatrix}1\\2\\0\\3\end{pmatrix}, \begin{pmatrix}2\\4\\1\\6\end{pmatrix}, \begin{pmatrix}2\\2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix}$.

M2. Let $V$ be a vector space, $\varphi \in V^*$ a linear functional and $\underline{w} \in V$ a fixed vector. Suppose that $\varphi(\underline{w}) \neq 0$.
(a) Show directly that $V = \operatorname{Ker}\varphi \oplus \operatorname{Span}(\underline{w})$.
(b) Show that the map $T : V \to V$ given by $T\underline{v} = \underline{v} - 2\frac{\varphi(\underline{v})}{\varphi(\underline{w})}\underline{w}$ is linear, and compute $T^2$.
(c) What are the eigenvalues of $T$? The eigenspaces? Find a basis of $V$ consisting of eigenvectors.

### Practice problems 2: calculating with inner products

M3. Let $S = \left\{ \begin{pmatrix}i\\0\\0\end{pmatrix}, \begin{pmatrix}1\\i+1\\1-2i\end{pmatrix}, \begin{pmatrix}0\\5+2i\\1+2i\end{pmatrix} \right\} \subset \mathbb{C}^3$.
(a) Calculate the 9 pairwise inner products of the vectors.
(b) Calculate the norms of the three vectors (recall that $\|\underline{x}\| = \sqrt{\langle \underline{x}, \underline{x} \rangle}$).

M4. Let $S = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix} \right\} \subset \mathbb{R}^3$.
(a) Verify that this is an orthonormal basis of $\mathbb{R}^3$.
(b) Find the coordinates of the vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}5\\6\\7\end{pmatrix}$ in this basis using the inner product.

M5. Using the standard ($L^2$) inner product on $C(-1,1)$ apply the Gram–Schmidt procedure to the following independent sequences:
(a) $\{1, x, x^2, x^3\}$ (in that order)
RMK Applying the Gram–Schmidt procedure to the full sequence $\{x^n\}_{n=0}^{\infty}$ yields the sequence of *Legendre polynomials* $P_n(x)$ (with a non-standard normalization).
(b) $\{x^3, x^2, x, 1\}$ (in that order)

RMK We can do the same with other inner products. Repeat part (a) with the inner products:
(c) (Hermit polynomials) $\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)g(x)e^{-x^2}\,dx$.
(d) (Laguerre polynomials) $\langle f, g \rangle = \int_0^{\infty} f(x)g(x)e^{-x}\,dx$.

## More on diagonalization

1. (a) Show that every $T \in \mathrm{End}(V)$ has a real eigenvalue if $V$ is a real vector space and $\dim_{\mathbb{R}} V$ is odd.
   (b) Define $T\colon \mathbb{R}[x]^{\leq 3} \to \mathbb{R}[x]^{\leq 3}$ by $(Tp)(x) = x^3 p(-1/x)$. Prove that $T$ has no real eigenvalues. (Hint: what is $T^2$?)
   (c) Define $T\colon \mathbb{C}[x]^{\leq 3} \to \mathbb{C}[x]^{\leq 3}$ by $(Tp)(x) = x^3 p(-1/x)$. Find the spectrum of $T$ and exhibit one eigenvector for each eigenvalue.

2. Let $V$ be a vector space, let $\{\lambda_i\}_{i=1}^r$ be *distinct* numbers, and let $T \in \mathrm{End}(V)$ satisfy $p(T) = 0$ where $p(x) = (x-\lambda_1)\cdots(x-\lambda_r) = \prod_{i=1}^r (x-\lambda_i)$.
   (a) Show that the spectrum of $T$ is contained in $\{\lambda_i\}_{i=1}^r$.
   (b) Fix $j$ and define an auxiliary map $R_j \in \mathrm{End}(V)$ by $R_j = \prod_{i\neq j}\left(\frac{T-\lambda_i}{\lambda_j-\lambda_i}\right)$. Show that $T \cdot R_j = \lambda_j R_j$.
   (c) Show by induction on $k$ that $T^k R_j = \lambda_j^k R_j$ for all $k \geq 0$.
   (d) Show that for any polynomial $q \in \mathbb{C}[x]$ we have an equality of linear maps $q(T)R_j = q(\lambda_j)R_j$ (on the left we compose the linear maps $q(T)$ and $R_j$; on the right we multiply the linear map $R_j$ by the scalar $q(\lambda_j)$).
   (**e) Show that $R_j$ is a projection.
   (f) Show that $\mathrm{Im}(R_j) = \mathrm{Ker}(T-\lambda_j)$.
   (**g) Show that $T$ is diagonable.

## Inner products

3. Find an orthonormal basis for the subspace $W^\perp \subset \mathbb{R}^4$ if $W = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}$.

4. The *trace* of a square matrix is the sum of its diagonal entries ($\mathrm{tr}A = \sum_{i=1}^n a_{ii}$).
   PRAC (MT2) Show that $\mathrm{tr}\colon M_n(\mathbb{R}) \to \mathbb{R}$ is a linear functional and that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B$, concluding that $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(A)$ if $S$ is invertible. On the other hand (**) find three 2x2 matrices $A, B, C$ such that $\mathrm{tr}(ABC) \neq \mathrm{tr}(BAC)$.
   (a) Show that $\langle A, B\rangle \stackrel{\text{def}}{=} \mathrm{tr}(A^t B)$ is an inner product on $M_n(\mathbb{R})$
   DEF For $A \in M_{m,n}(\mathbb{C})$, its *Hermitian conjuate* is the matrix $A^\dagger \in M_{n,m}(\mathbb{C})$ with entries $a_{ij}^\dagger = \overline{a_{ji}}$ (complex conjuguate).
   (d) Show that $\langle A, B\rangle \stackrel{\text{def}}{=} \mathrm{tr}\left(A^\dagger B\right)$ is a Hermitian product on $M_n(\mathbb{C})$.

## Extra credit: commuting transformations

P1. Fix a vector space $V$ and let $T, S \in \mathrm{End}(V)$ satisfy $TS = ST$.
   (a) Suppose that $T\underline{v} = \lambda\underline{v}$ for some $\lambda$ and $\underline{v} \in V$. Show that $T(S\underline{v}) = \lambda(S\underline{v})$.
   CONCLUSION Let $V_\lambda = \{\underline{v} \in V \mid T\underline{v} = \lambda\underline{v}\}$. Then $S(V_\lambda) \subset V_\lambda$.
   SUPP Let $A, B$ be invertible linear maps. Show that $AB = BA$ iff $ABA^{-1}B^{-1} = \mathrm{Id}$.
   DEF *An image of the discrete Heisenberg group* is a triple of invertible maps $A, B, Z \in \mathrm{End}(V)$ such that $ABA^{-1}B^{-1} = Z$ and such that $AZA^{-1}Z^{-1} = BZB^{-1}Z^{-1} = \mathrm{Id}$ ("$A, B$ commute with their commutator"). Fix such a triple for the rest of the problem.
   (*b) Let $\zeta$ be an eigenvalue of $Z$, and let $\lambda$ be an eigenvalue of the map $A\restriction_{V_\zeta}$ we bound in problem (a) (we set $V_\zeta = \mathrm{Ker}(Z-\zeta)$). Show that $\lambda\zeta$ is also an eigenvalue of $A\restriction_{V_\zeta}$ (hint: try doing something to the eigenvector).
   (c) Suppose $V$ is finite-dimensional. Show that we must have $\zeta^k = 1$ for some $k$.

(d) Compute $\det(Z\restriction_{V_\zeta})$ in two different ways to show that $\zeta^{\dim V_\zeta}=1$.

## Extra credit: norms

DEFINITION. Let $V$ be a real or complex vector space. A *norm* (="notion of length") on $V$ is a map $\|\cdot\|:V\to\mathbb{R}_{\geq0}$such that

(1) $\|a\underline{v}\|=|a|\,\|\underline{v}\|$ (that is, $3\underline{v}$ is three times as long as $\underline{v}$)
(2) $\|\underline{u}+\underline{v}\|\leq\|\underline{u}\|+\|\underline{v}\|$ ("triangle inequality")
(3) $\|\underline{v}\|=0$ iff $\underline{v}=\underline{0}$ (note that one direction follows from (1)).

P2. (Examples of norms)
(a) Show that $\|\underline{x}\|_\infty=\max_i|x_i|$ and $\|\underline{x}\|_1=\sum_i|x_i|$ are norms on $\mathbb{R}^n$ or $\mathbb{C}^n$.
(b) Show that $\|f\|_\infty=\max_{a\leq x\leq b}|f(x)|$ and $\|f\|_1=\int_a^b|f(x)|\,dx$ are norms on $C(a,b)$ (continuous functions on the interval $[a,b]$).
(c) (Sobolev norm) Show that $\|f\|_{H^1}^2=\int_a^b\left(|f(x)|^2+|f'(x)|^2\right)\mathrm{d}x$ defines a norm on $C^\infty(a,b)$ (Hint: this norm is associated to an inner product)

## Supplementary problem: $\ell^p$ norms

A. For $1\leq p<\infty$ and $\underline{x}\in\mathbb{C}^n$ define $\|\underline{x}\|_p=\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}$.
(a) Show that $\|\underline{x}\|=0$ iff $\underline{x}=\underline{0}$ and that $\|\alpha\underline{x}\|_p=|\alpha|\,\|\underline{x}\|_p$ for all scalars $\alpha$.
(b) Show that $\lim_{p\to\infty}\|\underline{x}\|_p=\|\underline{x}\|_\infty$.
RMK This justifies the notation from problem P2.

B Fix $p\in(1,\infty)$ and let $q\in(1,\infty)$ be defined by $\frac{1}{p}+\frac{1}{q}=1$ (we say the exponents $p,q$ are *dual*).
(a) Prove *Young's inequality*: for all $a,b\geq0$ we have $ab\leq\frac{a^p}{p}+\frac{b^q}{q}$.
*Hint*: Use the convexity of the function $f(t)=a^{(1-t)p}b^{tq}$, or direct calculus.
(b) Summing over the coordinates show for any $\underline{x},\underline{y}$ that $|\sum_{i=1}^n x_i\overline{y_i}|\leq\frac{1}{p}\|\underline{x}\|_p^p+\frac{1}{q}\|\underline{y}\|_q^q$.
(c) Replacing $\underline{x}$ with $\frac{x}{\|\underline{x}\|_p}$ and $\underline{y}$ with $\frac{y}{\|\underline{y}\|_q}$ and using the scaling behaviour from part A(a), prove *Hölder's inequality*

$$\left|\langle\underline{y},\underline{x}\rangle\right|=\|\underline{x}\|_p\|\underline{y}\|_q\,.$$

(d) Check that the inequality also holds in the extreme cases $p=1,q=\infty$ and $p=\infty,q=1$ (these exponents are dual if we interpret $\frac{1}{\infty}=0$).
(e) Show that $\|\underline{x}\|_p=\max\left\{\langle\underline{y},\underline{x}\rangle:\ \|\underline{y}\|_q=1\right\}$.
*Hint*: Choose $y_i$ so that $x_i\overline{y_i}=c|x_i|^p$ for a positive constant $c$ chosen so that $\|\underline{y}\|_q=1$.
(f) Show that $\|\underline{x}+\underline{x}'\|_p\leq\|\underline{x}\|_p+\|\underline{x}'\|_p$ for all $\underline{x},\underline{x}'$.
*Hint*: $\langle\underline{y},\underline{x}+\underline{x}'\rangle=\langle\underline{y},\underline{x}\rangle+\langle\underline{y},\underline{x}'\rangle$.

C. DEF Let $\ell^p=\left\{\underline{a}\in\mathbb{C}^{\mathbb{N}}\mid\sum_{i=1}^\infty|a_i|^p<\infty\right\}$ (read: "ell-p" be the space of $p$-summable sequences.
(a) Use scaling and Minkowski's inequality (for the partial sums of the series) to show that $\ell^p$ is a subspace of $\mathbb{C}^{\mathbb{N}}$.
(b) Show that $\|\underline{a}\|_p=\left(\sum_{i=1}^\infty|a_i|^p\right)^{1/p}$ is a norm on $\ell^p$.
(c) Show that $\ell^p\subset\ell^q$ if $p\leq q$.