Math 100 – SOLUTIONS TO WORKSHEET 9
THE CHAIN RULE; INVERSE FUNCTIONS

1. The chain rule

(1) Write the function as a composition and then differentiate.

(a) e^{3x}

Solution: This is $f(g(x))$ where $g(x) = 3x$ and $f(y) = e^x$. The derivative is thus

$$e^{3x} \cdot \frac{\mathrm{d}(3x)}{\mathrm{d}x} = 3e^{3x}\,.$$

(b) $\sqrt{2x+1}$

Solution: This is $f(g(x))$ where $g(x) = 2x+1$ and $f(y) = \sqrt{y}$. Thus

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = f'(g(x))g'(x) = \frac{1}{2\sqrt{g}} \cdot 2 = \frac{1}{\sqrt{2x+1}}\,.$$

(c) (Final, 2015) $\sin(x^2)$

Solution: This is $f(g(x))$ where $g(x) = x^2$ and $f(y) = y^2$. The derivative is then

$$\cos(x^2) \cdot 2x = 2x\cos(x^2)\,.$$

(d) $(7x + \cos x)^n$.

Solution: This is $f(g(x))$ where $g(x) = 7x + \cos x$ and $f(y) = y^n$. The derivative is thus

$$n\,(7x + \cos x)^{n-1} \cdot (7 - \sin x)\ .$$

(2) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

Solution: By the chain rule, $f'(x) = g'(2\sin x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(2\sin x) = 2g'(2\sin x)\cos x$. In particular,

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) &= 2g'\left(2\sin\frac{\pi}{4}\right)\cos\frac{\pi}{4} = 2g'\left(2\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} \\ &= 2\sqrt{2} \cdot \frac{\sqrt{2}}{2} = 2. \end{aligned}$$

(3) Differentiate

(a) $7x + \cos(x^n)$

Solution: We apply linearity and then the chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(7x + \cos(x^n)\right) &= \frac{\mathrm{d}(7x)}{\mathrm{d}x} + \frac{\mathrm{d}\cos(x^n)}{\mathrm{d}x} \\ &= 7 + \frac{\mathrm{d}\cos(x^n)}{\mathrm{d}(x^n)} \cdot \frac{\mathrm{d}(x^n)}{\mathrm{d}x} \\ &= 7 - \sin(x^n) \cdot nx^{n-1}\,. \end{aligned}$$

(b) $e^{\sqrt{\cos x}}$

Solution: We repeatedly apply the chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{\cos x}} &= e^{\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\cos x} \\ &= e^{\sqrt{\cos x}}\frac{1}{2\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\cos x \\ &= -e^{\sqrt{\cos x}}\frac{\sin x}{2\sqrt{\cos x}}\,. \end{aligned}$$

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(c) (Final 2012) $e^{(\sin x)^2}$

Solution: By the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\sin x)^2}\right) &= e^{(\sin x)^2}\frac{\mathrm{d}}{\mathrm{d}x}\left((\sin x)^2\right)\\ &= e^{(\sin x)^2} 2\sin x \frac{\mathrm{d}}{\mathrm{d}x}\sin x\\ &= e^{(\sin x)^2} 2\sin x\cos x\\ &= e^{(\sin x)^2}\sin(2x).\end{aligned}$$

(4) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Solution: Applying the chain rule we have $f'(g(x)) \cdot g'(x) = 3x^2$. Plugging in $x = 4$ we get $5g'(4) = 3 \cdot 4^2$ and hence $g'(4) = \frac{48}{5}$.

2. Inverse functions

(5) Find the function inverse to $y = x^7 + 3$.

Solution: If $y = x^7 + 3$ then $x^7 = y - 3$ so $x = (y-3)^{1/7}$, and the inverse function is $\boxed{y = (x-3)^{1/7}}$.

(6) Does $y = x^2$ have an inverse?

Solution: Not on its full domain (not single-valued), yes on $[0, \infty)$.

(7) Consider the function $y = \sqrt{x-1}$ on $x \geq 1$.

(a) Find the inverse function, in the form $x = g(y)$.

Solution: If $y = \sqrt{x-1}$ then $y^2 = x - 1$ so $\boxed{x = y^2 + 1}$.

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}x}{\mathrm{d}y}$ and calculate their product.

Solution: We have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x-1}}$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = 2y$. Their product is $\frac{2y}{2\sqrt{x-1}} = \frac{y}{\sqrt{x-1}} = 1$ since $y = \sqrt{x-1}$ along the curves.

(8) Let $f(x) = \log x$. Apply the chain rule to the formula $f(e^y) = y$ to get a formula for $f'(e^y)$, and use that to determine the derivative of the logarithm.

Solution: We differentiate $f(e^y) = y$ with respect to y to get $f'(e^y) \cdot e^y = 1$ so $f'(e^y) = \frac{1}{e^y}$. Now let $e^y = x$ to get

$$\boxed{(\log x)' = \frac{1}{x}}.$$

(9) Let $f(x) = x^3 + 5x$. Find $f^{-1}(6)$ and $\left(f^{-1}\right)'(6)$.

Solution: We note that $f(1) = 6$ so $f^{-1}(6) = 1$. Also $\left(f^{-1}\right)'(6) = \frac{1}{f'(1)}$. Now $f'(x) = 3x^2 + 5$ so

$$\boxed{\left(f^{-1}\right)'(6) = \frac{1}{8}}.$$