Math 100 – SOLUTIONS TO WORKSHEET 7
DIFFERENTIATION RULES

1. The product and quotient rules

(1) Differentiate
  (a) $f(x) = 6x^\pi + 2x^e - x^{7/2}$
  Solution: This is a linear combination of power laws so $f'(x) = 6\pi x^{\pi-1} + 2ex^{e-1} - \frac{7}{2}x^{5/2}$.
  (b) (Final, 2016) $g(x) = x^2 e^x$ (and then also $x^a e^x$)
  Solution: Applying the product rule we get $\frac{dg}{dx} = \frac{d(x^2)}{dx} \cdot e^x + x^2 \cdot \frac{d(e^x)}{dx} = (2x + x^2)e^x = x(x+2)e^x$, and in general

$$\frac{d}{dx}(x^a e^x) = ax^{a-1}e^x + x^a e^x = x^{a-1}(x+a)e^x\,.$$

  (c) (Final, 2016) $h(x) = \frac{x^2+3}{2x-1}$
  Solution: Applying the quotient rule the derivative is $\frac{2x\cdot(2x-1)-(x^2+3)\cdot 2}{(2x-1)^2} = \frac{4x^2-2x-2x^2-6}{(2x-1)^2} = 2\frac{x^2-x-3}{(2x-1)^2}$.
  (d) $\frac{x^2+A}{\sqrt{x}}$
  Solution: We write the function as $x^{3/2} + Ax^{-1/2}$ so its derivative is $\frac{3}{2}x^{1/2} - \frac{A}{2}x^{-3/2}$.

(2) Let $f(x) = \frac{x}{\sqrt{x}+A}$. Given that $f'(4) = \frac{3}{16}$, give a quadratic equation for A.
  Solution: $f'(x) = \frac{1\cdot\left(\sqrt{x}+A\right) - x\left(\frac{1}{2}x^{-1/2}\right)}{\left(\sqrt{x}+A\right)^2} = \frac{\sqrt{x}+A-\frac{1}{2}\sqrt{x}}{\left(\sqrt{x}+A\right)^2} = \frac{\frac{1}{2}\sqrt{x}+A}{\left(\sqrt{x}+A\right)^2}$. Plugging in $x = 4$ we have

$$\frac{3}{16} = f'(4) = \frac{1+A}{(2+A)^2}$$

so we have

$$3(2+A)^2 = 16(1+A)$$

that is

$$3A^2 + 12A + 12 = 16 + 16A$$

that is

$$3A^2 - 4A - 4 = 0\,.$$

In fact this gives $A = -\frac{2}{3}, 2$.

(3) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$. Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.
  Solution: $(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3\cdot 2 + 1\cdot 4 = 10$.

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2} = \frac{3\cdot 2 - 1\cdot 4}{2^2} = \frac{1}{2}\,.$$

2. The tangent line

(1) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.
  Solution: $f'(x) = \frac{1}{2\sqrt{x}}$, so the slope of the line is $f'(4) = \frac{1}{4}$, and the equation for the line line itself is $y - 2 = \frac{1}{4}(x-4)$ or $y = \frac{1}{4}(x-4) + 2$ or $y = \frac{1}{4}x + 1$.

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(2) Let $f(x) = \frac{g(x)}{x}$, where $g(x)$ is differentiable at $x = 1$. The line $y = 2x - 1$ is tangent to the graph $y = f(x)$ at $x = 1$. Find $g(1)$ and $g'(1)$.

Solution: At $x = 1$ the line meets the graph of $y = f(x)$ so $2 \cdot 1 - 1 = 1 = f(1) = \frac{g(1)}{1}$ and we concldue that $g(1) = 1$. The slope of the line there is 2, so $f'(1) = 2$. Since we have

$$f'(x) = \frac{xg'(x) - g(x)x}{x^2}$$

we have $2 = f'(1) = g'(1) - g(1)$ so $g'(1) = 2 + g(1) = 3$.

(3) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

Solution: The line has slope 4 and meets the curve at $(1, 6)$. The last two functions don't evaluate to 6 at 1.We differentiate the first three.

$$\begin{aligned}
\frac{d}{dx}|_{x=1}\left(x^3 + 2x^2 + 3x\right) &= \left(3x^2 + 4x + 3\right)|_{x=1} = 10 \\
\frac{d}{dx}|_{x=1}\left(x^2 + 3x + 2\right) &= (2x + 3)\,|_{x=1} = 5 \\
\frac{d}{dx}|_{x=1}\left(2\sqrt{x+3} + 2\right) &= \left(\frac{2}{2\sqrt{x+3}}\right)|_{x=1} = \frac{1}{2}\,.
\end{aligned}$$

The answer is "none of the above".

(4) Find the lines of slope 3 tangent the curve $y = x^3 + 4x^2 - 8x + 3$.

Solution: $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 8x - 8$, so the line tangent at (x, y) has slope 3 iff $3x^2 + 8x - 8 = 3$, that is iff $3(x^2 - 1) + 8(x - 1) = 0$. We can factor this as $(x - 1)(3x + 11) = 0$ so the x-coordinates of the points of tangency are $1, -\frac{11}{3}$ and the lines are:

$$\begin{aligned}
y &= 3(x - 1) \\
y &= 3(x + \frac{11}{3}) + \left(\left(\frac{11}{3}\right)^3 + 4\left(\frac{11}{3}\right)^2 - 8\left(\frac{11}{3}\right) + 3\right)\,.
\end{aligned}$$

(5) The line $y = 5x + B$ is tangent to the curve $y = x^3 + 2x$. What is B?

Solution: At the point (x, y) the curve has slope $\frac{dy}{dx} = 3x^2 + 2$, so the curve has slope 5 at the points where $x = \pm 1$, that is the points $(-1, -3)$ and $(1, 3)$. The line needs to meet the curve at the point, so there are two solutions:

$$\begin{aligned}
y &= 5x + 2 \qquad \text{(tangent at } (-1, -3)) \\
y &= 5x - 2 \qquad \text{(tangent at } (1, 3))
\end{aligned}$$