Math 100 – SOLUTIONS TO WORKSHEET 6
THE DERIVATIVE

1. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

(1) Find $f'(a)$ if
 (a) $f(x) = x^2$, $a = 3$.
 Solution: $\lim_{h\to 0} \frac{(3+h)^2-(3)^2}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h) = 6$.
 (b) $f(x) = \frac{1}{x}$, any a.
 Solution: $\lim_{h\to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{a-(a+h)}{a(a+h)}\right) = \lim_{h\to 0} \frac{-h}{h\cdot a(a+h)} = -\lim_{h\to 0} \frac{1}{a(a+h)} = -\frac{1}{a^2}$.
 (c) $f(x) = x^3 - 2x$, any a (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).
 Solution: We have

$$\begin{aligned}\frac{(a+h)^3 - 2(a+h) - a^3 + 2a}{h} &= \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2a - 2h - a^3 + 2a}{h} \\ &= \frac{3a^2h + 3ah^2 + h^3 - 2h}{h} \\ &= 3a^2 - 2 + 3ah + h^2 \xrightarrow[h\to 0]{} 3a^2 - 2\,.\end{aligned}$$

(2) Express the limits as derivatives: $\lim_{h\to 0} \frac{\cos(5+h)-\cos 5}{h}$, $\lim_{h\to 0} \frac{\sin x}{x}$
 Solution: These are the derivative of $f(x) = \cos x$ at the point $a = 5$ and of $g(x) = \sin x$ at the point $a = 0$.

(3) (Final, 2015) Is the function

$$f(x) = \begin{cases} \sqrt{1+x^2} - 1 & x \le 0 \\ x^2 \cos\frac{1}{x} & x > 0 \end{cases}$$

differentiable at $x = 0$?
 Solution: We have $f(0) = \sqrt{1+0} - 1 = 0$, so we'd have $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} = \lim_{x\to 0} \frac{f(x)}{x}$ provided the limit exists, and since we have different expresions for $f(x)$ on both sides of 0 we compute the limit as two one-sided limits. On the left we have

$$\begin{aligned}\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{\sqrt{1+x^2}-1}{x} &= \lim_{x\to 0^-} \frac{(1+x^2)-1}{x\left(\sqrt{1+x^2}+1\right)} \\ = \lim_{x\to 0^-} \frac{x^2}{x\left(\sqrt{1+x^2}+1\right)} &= \lim_{x\to 0^-} \frac{x}{\sqrt{1+x^2}+1} = 0\,.\end{aligned}$$

Alternatively, we could recognize the limit

$$\lim_{x\to 0^-} \frac{\sqrt{1+x^2} - \sqrt{1-0^2}}{x}$$

as giving the derivative of $f(x) = \sqrt{1+x^2}$ at $x = 0$. Using differentiation rules (to be covered later in the course) we know that $\left[\frac{d}{dx}\sqrt{1+x^2}\right]_{x=0} = \left[\frac{2x}{2\sqrt{1+x^2}}\right]_{x=0} = 0$ and it would again follow that

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$\lim_{x\to 0^-} \frac{f(x)}{x} = 0$.
On the right we have

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2\cos\frac{1}{x}}{x} = \lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right) = 0$$

by the squeeze theorem (we have $-x \le x\cos\frac{1}{x} \le x$ for all $x > 0$ and $\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0$). Since both limits are 0 the function f is differentiable at $x = 0$ and $f'(0) = 0$.

2. Linear combinations; power laws

(4) Let $g(y) = Ay^{5/2} + y^2$. Suppose that $g'(4) = 0$. What is A?

Solution: Differentiating we find $g'(y) = \frac{5}{2}Ay^{3/2} + 2y$, so $0 = g'(4) = \frac{5}{2}A\cdot 4^{3/2} + 2\cdot 4 = \frac{5}{2}\cdot A\cdot 8 + 8$. This means: $20A + 8 = 0$ so $A = -\frac{2}{5}$.

(5) Find the *second* derivative of $5t + 3\sqrt{t}$

Solution: $t'' = 0$ and $(\sqrt{t})'' = \left(\frac{1}{2}t^{-1/2}\right)' = -\frac{1}{4}t^{-3/2}$ so by linearity the second derivative is $-\frac{3}{4}t^{-3/2}$.

(6) Differentiate $f(x) = \frac{5x^3-2x+1}{\sqrt{x}}$.

Solution: Write $f(x) = 5x^{5/2} - 2x^{1/2} + x^{-1/2}$ and then $f'(x) = \frac{25}{2}x^{3/2} - x^{-1/2} - \frac{1}{2}x^{-3/2}$.

(7) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

Solution: $f'(x) = \frac{1}{2\sqrt{x}}$, so the slope of the line is $f'(4) = \frac{1}{4}$, and the equation for the line itself is $y - 2 = \frac{1}{4}(x-4)$ or $y = \frac{1}{4}(x-4) + 2$ or $y = \frac{1}{4}x + 1$.