21. Antiderivatives (25/11/2021)

Goals.

(1) Idea of inverse operation
(2) Antiderivatives by massaging
(3) Antiderivatives of sums

Last Time. **l'Hôpital's rule**

Have $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ $\boxed{\text{IF}}$ we first check $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ (or both infinite)

Have to check this

Holds for limits at ∞,
for limits in extended sense
($\lim = \infty$, $\lim = -\infty$)

(1) Sometimes need to get limit to have this form

($(x^2+1)e^{-x} = \frac{x^2+1}{e^x}$; $(2x+1)^{\frac{1}{\sin x}} = e^{\frac{\log(2x+1)}{\sin x}}$,

$x^x = e^{x\log x} = e^{\frac{\log x}{1/x}}$).

Today: anti-derivatives

So far: given F computed f s.t. $f = F'$!

Today: given f find F s.t. $F' = f$

Math 100 – WORKSHEET 21
ANTIDERIVATIVES

1. Warmup: inverse operations

(1) (Multiplication)
(a) Calculate: $7 \times 8 =$ 56
(b) Find (some) integral a, b such that $ab = 15$.

$15 = 3 \times 5$

also 1×15, 5×3, 15×1

(1) multiplication easier than factoring $\Rightarrow$ easy to check answers
(2) multiplication has one answer, reverse multiplication = factoring can have several

(2) (Trig functions)
(a) Calculate: $\sin \frac{\pi}{3} =$ $\sqrt{3}/2$
(b) Find all θ such that $\sin \theta = 1$.

$\ldots, \frac{\pi}{2} - 2\pi, \frac{\pi}{2}, \frac{\pi}{2} + 2\pi, \frac{\pi}{2} + 4\pi, \ldots$

Write the set as $\left\{ \frac{\pi}{2} + 2\pi k \mid k \in \mathbb{Z} \right\}$

(1)
(2) as before

structure to solutions
(can find all solutions from one)

Date: 25/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Say $f'(x) = 1$

let $g(x) = f(x) - x$, then $g'(x) = 1 - 1 = 0$

If $g'(x) \equiv 0$ then g is constant:

Pf: If $a < b$, $\dfrac{g(b) - g(a)}{b - a} = g'(c) = 0$ for some $a < c < b$ (by MVT)

so $g(b) = g(a)$.

so $g(x) \equiv C$ for some constant C

so $f(x) = x + C$

(in general, if $f'(x) = h(x)$ then the general solution to $f' = h$ is $f + C$)

(3) Simple differentiation

(a) Find one f such that $f'(x) = 1$.

Take $f(x) = x$

(also $x+1$, $x+7$, $x-\pi$, ...

(b) Find *all* such f.

$\{ x + c \mid c \in \mathbb{R} \text{ constant} \}$ (usually just write $f(x) = x + c$ call it "the general solution")

(c) Find the f such that $f(7) = 3$.

Have $f(x) = x + c$. To have $f(7) = 3$ need $7 + c = 3$

so $c = -4$, i.e. $\boxed{f(x) = x - 4}$ is that function.

particular solution (such that $f(7) = 3$)

Summary: (1) anti-differentiation harder: need to "guess" a solution; no rules. (But easy to check answer)

(2) once we have one solution, the "general solution" is given by shifts: $+ C$.

(3) ~~...~~ If we have additional conditions can find the particular solution that satisfies them by setting up an equation for C.

2. ANTIDIFFERENTIATION BY MASSAGING

(4) Find f such that $f'(x) = 2x^3$.

Notice x^3, then x^4 is about right: $\frac{d}{dx}x^4 = 4x^3$

we're off by a factor of 2:

$$\frac{d}{dx}\left(\frac{1}{2}x^4\right) = \frac{1}{2}\cdot 4\cdot x^3 = 2x^3 \quad ✓$$

(general solution is ~~[illegible]~~ $\frac{1}{2}x^4 + C$)

(5) Find f such that $f'(x) = -\frac{1}{x}$.

<u>Naive</u>: We know $(\log x)' = \frac{1}{x}$, so try $f(x) = -\log x$

<u>Problem</u>: $-\log x$ has wrong domain: only defined if $x>0$. ($\frac{1}{x}$ defined for all $x \neq 0$)

<u>**correct:**</u> $(\log|x|)' = \frac{1}{x}$ so $f(x) = -\log|x|$ works.

(6) Find all f such that $f'(x) = \cos 3x$.

We know $(\sin x)' = \cos x$; try $\sin 3x$: $(\sin 3x)' = 3\,\cos(3x)$

So $\left(\frac{1}{3}\sin 3x\right)' = \cos(3x)$,

The general solution is $\frac{1}{3}\sin(3x) + C$

["Find the most general anti-derivative of $\cos(3x)$"]

Fact: $\frac{d}{dx}(\log|x|) = \frac{1}{x}$ for all $x \neq 0$

How to discover this?

suppose we want to solve $f'(x) = \frac{1}{x}$ for $x<0$.

idea: let $y = -x$ then $y>0$, $f'(x) = -\frac{1}{y}$

$\frac{d}{dy}(\log y) = \frac{1}{y}$, $\frac{dy}{dx} = -1$ so $\frac{d(\log y)}{dx} = \frac{d(\log y)}{dy} \cdot \frac{dy}{dx}$

$= \frac{1}{y} \cdot -1 = -\frac{1}{y}$

so $f(x) = \log y = \log(-x) = \log|x|$ if $x<0$.

has $f'(x) = \frac{1}{x}$

Thus $\boxed{\log|x|}$ solves problem (\$)

If I need to compute $\lim_{x\to-\infty} f(x)$

we often set $y = -x$, take limit as $y \to \infty$

$x = -y$

3. Combinations

(7) (Final, 2015) Find a function $f(x)$ such that $f'(x) = \sin x + \frac{2}{\sqrt{x}}$ and $f(\pi) = 0$.

massaging to find one solution

$(\cos x)' = -\sin x$, $(x^{1/2})' = \frac{1}{2}x^{-1/2}$ so $(-\cos x + 4x^{\frac{1}{2}})' = \sin x + \frac{2}{\sqrt{x}}$

Thus $f(x) = -\cos x + 4\sqrt{x} + C$ for some C ← general solution

To find C, we have $f(\pi) = -\cos \pi + 4\sqrt{\pi} + C = 0$

so $1 + 4\sqrt{\pi} + C = 0$ so $C = -4\sqrt{\pi} - 1$

so $\boxed{f(x) = -\cos x + 4\sqrt{x} - 4\sqrt{\pi} - 1}$ ← particular solution with $f(\pi) = 0$

(8) (Final, 2016) Find the general antiderivative of $f(x) = e^{2x+3}$.

$\frac{d}{du}(e^u)' = e^u$ so try $(e^{2x+3})' = e^{2x+3} \cdot 2$

so $\frac{1}{2}e^{2x+3}$ works, and the general solution

is $\boxed{\frac{1}{2}e^{2x+3} + C}$

(9) Find f such that $f'(x) = \frac{6x^4-2x-2}{x^2}$.

this is the same as $f'(x) = 6x^2 - \frac{2}{x} - \frac{2}{x^2}$

$(x^3)' = 3x^2$, $(\log|x|)' = \frac{1}{x}$, $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

so $f(x) = 2x^3 - 2\log|x| + 2\frac{1}{x}$ works.

(divide, massage pieces, put together)

(10) Find f such that $f'(x) = 2x^{1/3} - x^{-2/3}$ and $f(1000) = 5$.

$(x^{4/3})' = \frac{4}{3}x^{1/3}$, $(x^{1/3})' = \frac{1}{3}x^{-2/3}$

so $f(x) = \frac{3}{2}x^{4/3} - 3x^{1/3} + c$

$5 = f(1000) = \frac{3}{2} \cdot 10^4 - 30 + c$ so $c = -15{,}000 + 35$.

(11) Find f such that $f''(x) = \sin x + \cos x$, $f(0) = 0$ and $f'(0) = 1$.