17. Optimization (9/11/2021)

Goals.

(1) Problem solving
(2) Examples

Last Time. Finding extrema of functions:

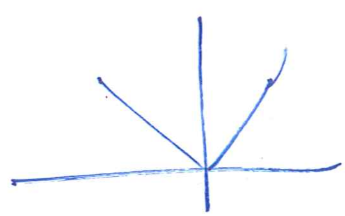
If f is cts on $[a,b]$ then absolute min/max of f on the interval exist, occur at one of:

(1) critical pts: $f'(x_0)=0$
(2) singular pts: $f'(x_0)$ DNE
(3) end pts: a, b

(really is about closed intervals: $\frac{1}{x(1-x)}$ has no max on $(0,1)$, neither does $x+3$)

Q: why need singular pts?

Ex: $f(x)=|x|$ on $[-1,1]$.



min at $x_0=0$

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open interval: (a,b) : $\{x: a<x<b\}$ ← excludes endpoints

closed interval: $[a,b]$: $\{x: a \le x \le b\}$ ← includes endpoints

Example 4: We have a square sheet of cardboard (12 cm × 12 cm); we'd like to cut the corners and fold the cardboard into a box, what is the largest volume we can get?

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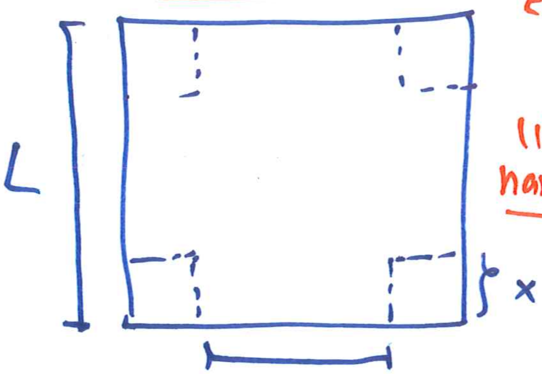
picture



(0) understand problem

(1) names: let x be the length of the cuts (in cm)
V be the volume of the resulting box.
L be the side of the square

$0 \le x \le L/2 = 6$

(2) relations: base of the box will be a square with side $L-2x$, height of the box is x

so the volume of the box is $V(x) = (L-2x)^2 \cdot x$

(3) calculus: $V(x)$ is diff & cts on $[0, L/2]$,

$$V'(x) = 2(-2)(L-2x)\cdot x + (L-2x)^2$$
$$= (L-2x)(-4x + L - 2x) = (L-2x)(L-6x)$$

so it has a critical pt at $x = L/6$ (& at endpoint $L/2$)

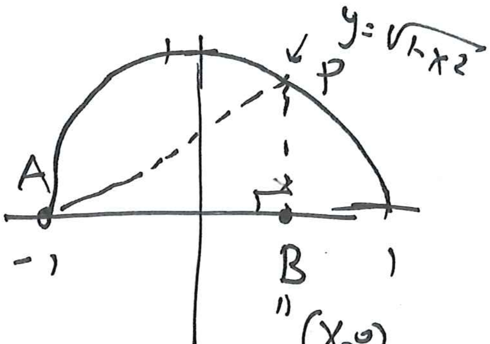
$$V(0) = V(L/2) = 0, \quad V(L/6) = (L - L/3)^2 \cdot \frac{L}{6} = L^3 \cdot \frac{2}{27}$$

so maximum is at $x = L/6$, $V_{max} = \frac{2}{27} L^3$.

(4) endgame: The largest volume we can get is $\frac{2}{27} L^3 = 128\ cm^3$ if we cut at $x = L/6 = 2cm$.

Math 100 – WORKSHEET 17
OPTIMIZATION

(1) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1-x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of this triangle?



let $B = (x, 0)$
then $P = (x, \sqrt{1-x^2})$
so area of the triangle is

$$A(x) = \frac{1}{2}\cdot\underbrace{(x-(-1))}_{\text{base}}\cdot\underbrace{\sqrt{1-x^2}}_{\text{height}} = \frac{1}{2}(x+1)\sqrt{1-x^2}$$

$-1 \le x \le 1$

Now A is cts on $[-1,1]$ (defined by formula)

$A(-1) = \frac{1}{2}\cdot 0 \cdot 0 = 0$, $A(1) = \frac{1}{2}\cdot 2 \cdot 0 = 0$

In the interior

$$A'(x) = \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(x+1)\frac{-2x}{2\sqrt{1-x^2}}$$

$$= \frac{1}{2\sqrt{1-x^2}}\left((1-x^2)+(x+1)(-2x)\right) = \frac{1-3x^2-2x}{2\sqrt{1-x^2}}$$

singularities at ± 1, critical pt at $\frac{-2\pm\sqrt{2^2+12}}{6} = \frac{-2\pm\sqrt{16}}{6} = -1, \frac{1}{3}$
(possibly) possibly

Date: 9/11/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$A(\frac{1}{3}) = \frac{1}{2}\cdot\frac{4}{3}\sqrt{1-\frac{1}{9}} = \frac{4\sqrt{2}}{9}$, so largest possible area is $\frac{4\sqrt{2}}{9}$
(obtained at $x = \frac{1}{3}$)

$$A(x) = \frac{1}{2}(x+1)\sqrt{1-x^2} = \frac{1}{2}(1+x)^{3/2}(1-x)^{1/2}$$

$$A'(x) = \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(x+1)\frac{-2x}{2\sqrt{1-x^2}}$$

$$= \frac{(1-x^2) - x(x+1)}{2\sqrt{1-x^2}} = \frac{1-x-2x^2}{2\sqrt{1-x^2}} = \frac{(x+1)(-2x+1)}{2\sqrt{1-x^2}}$$

$$= \frac{\sqrt{x+1}}{2\sqrt{1-x}}(1-2x)$$

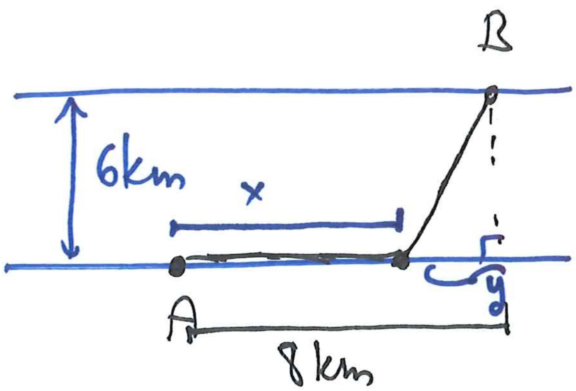
<u>see:</u> singular pt at $x = 1$

critical pt at $x = \frac{1}{2}$, $A\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$

maximum at $x = \frac{1}{2}$, area is $\frac{3\sqrt{3}}{8}$.

bank = edge of river.

(2) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?



Construct a path from A to B by building a road of length x and then a bridge

in terms of x, $0 \le x \le 8$, the cost is

$$C(x) = 20x + 40\sqrt{6^2 + (8-x)^2}$$

(in terms of y, $0 \le y \le 8$, cost is

$$C(y) = 20(8-y) + \sqrt{6^2 + y^2} \cdot 40$$

($x = 0$, $y = 8$ is bridge from A to B)

C is cts on $[0, 8]$, $C(0) = 40\sqrt{6^2 + 8^2} = 400$

$C(8) = 160 + 40\sqrt{6^2} = 400$

$$C'(x) = 20 + \frac{40 \cdot (2x - 16)}{2\sqrt{x^2 - 16x + 100}}$$

so $C'(x) = 0$ if

$-40\sqrt{x^2 - 16x + 100} = 40(2x - 16) \Rightarrow -(x^2 - 16x + 100) = 4x^2 - 64x + 64$

$$20 + \frac{90(2x-16)}{2\sqrt{x^2-16x+100}} = 0 \quad \text{so} \quad \frac{90(2x-16)}{2\sqrt{}} = -20$$

$$\text{so} \quad 90(2x-16) = -90\sqrt{}$$