# 8. Exponential and Trig functions (7/10/2021)

Goals.

(1) Exponential functions
(2) Trig functions: the definition; their derviatives

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Last Time. Diff rules: $(fg)' \neq f'g'$

$(af+bg)' = af' + bg'$, $(fg)' = f'g + fg'$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$. + deducing those from linear approximation.

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Tangent line: $y = f'(a)(x-a) + f(a)$

Determined by/determines: (1) $f'(a)$ (2) $f(a)$

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Idea: If we don't know the value of some quantity, give it a name, use that to calculate.

## Exponential functions

Have the form $f(x) = a^x$, call $a$ the base $\boxed{a>0}$

Then:

$$a^{x+y} = a^x \cdot a^y, \quad (a^x)^y = a^{xy}, \quad a^{-x} = \frac{1}{a^x}$$

$$(ar)^x = a^x \cdot r^x.$$

warning: $a^{(x^y)} \neq (a^x)^y$, $\quad a^{x^y} = a^{(x^y)}$

(cf. $2 \cdot 3 \cdot 4 = (2 \cdot 3) \cdot 4$)

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With it comes logarithm: $x = a^{\log_a x}$.

$$\log_a(xy) = \log_a x + \log_a y, \quad \log_a(x^y) = y \log_a x,$$

$$\log_a\left(\frac{1}{x}\right) = -\log_a x, \quad \log_r x = \frac{\log_a x}{\log_a r}$$

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In Math, $\log = \log_e$, $e$ TBD

(elsewhere, sometimes $\log = \log_{10}$ or $\log = \log_2$)

then need $\ln = \log_e$

here $\log_{10} x = \frac{\log x}{\log 10}$, $\log_2 x = \frac{\log x}{\log 2}$

# What is $\frac{d}{dx} q^x$ ?

By definition, it is $\lim\limits_{h\to 0} \frac{q^{x+h} - q^x}{h} = \lim\limits_{h\to 0} \frac{q^x q^h - q^x}{h} =$

$$= \lim_{h\to 0} q^x \frac{q^h - q^0}{h} = q^x \underbrace{\lim_{h\to 0} \frac{q^h - q^0}{h}}_{\left[\frac{dq^x}{dx}\right]_{x=0}}$$

Write: $L(q) = \lim\limits_{h\to 0} \frac{q^h - 1}{h} = \left[\frac{dq^x}{dx}\right]_{x=0}$

$L(2) \approx .693\ldots$

$L(3) \approx 1.013\ldots$

Conclusion: $\frac{d}{dx} q^x = L(q) \cdot q^x$

**Def:** $e$ is the number s.t. $L(e) = 1$

(i.e. s.t. $(e^x)' = e^x$)

**Fact:** $e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$

$\approx 2.718281828\ldots$

**Check:** $L(qr) = L(q) + L(r)$

$\Rightarrow L(q) = \log q$ ; $\boxed{(q^x)' = (\log q) \cdot q^x}$

Math 100 – WORKSHEET 8
EXPONENTIAL AND TRIG FUNCTIONS

# 1. Exponentials

(1) Simplify

(a) $(e^5)^3$, $(2^{1/3})^{12}$, $\cancel{7^{3-5}}$. $7^3/7^5$

$(e^5)^3 = e^{15}$; $(2^{1/3})^{12} = 2^{\frac{1}{3}\cdot 12} = 2^4 = 16$, $\frac{7^3}{7^5} = 7^{3-5} = 7^{-2} = \frac{1}{49}$.

(b) $\log(10e^5)$, $\log(3^7)$.

$\log(10e^5) = \log 10 + \log(e^5) = \log(10) + 5$

$\log(3^7) = 7\log 3$

(2) Differentiate:

(a) $10^x$

$(10^x)' = (\log 10)\cdot 10^x$

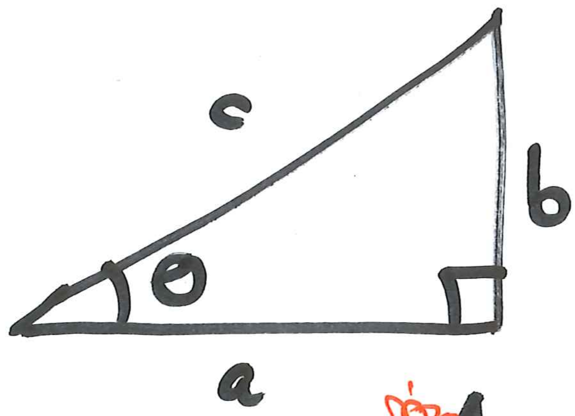
(b) $\frac{5\cdot 10^x + x^2}{3^x+1}$ quotient rule

$$\left(\frac{5\cdot 10^x + x^2}{3^x+1}\right)' = \frac{(5\cdot 10^x + x^2)'(3^x+1) - (5\cdot 10^x + x^2)(3^x+1)'}{(3^x+1)^2} =$$

$$= \frac{(5\cdot(\log 10)\cdot 10^x + 2x)(3^x+1) - (5\cdot 10^x + x^2)(\log 3)\cdot 3^x}{(3^x+1)^2}$$

*Date*: 7/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

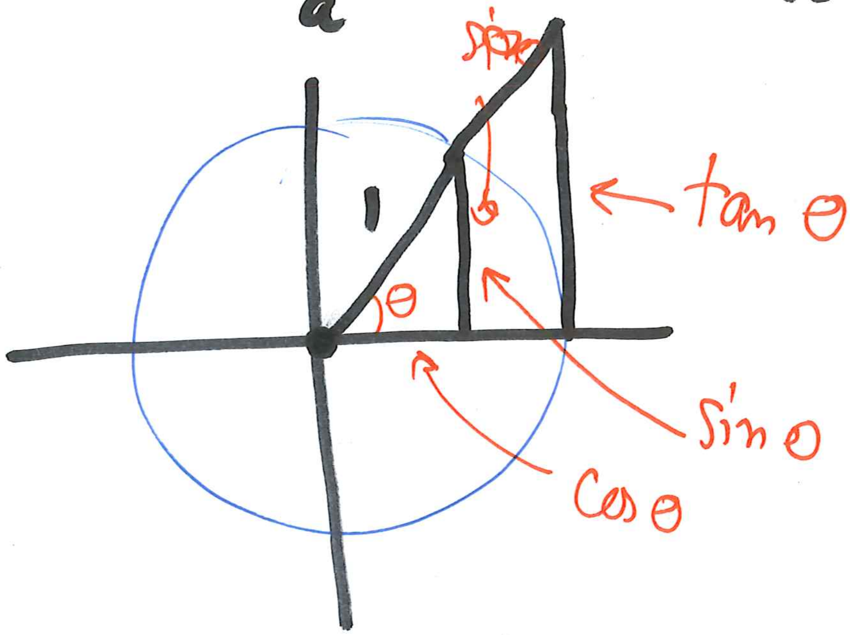
# Trig functions




$\sin\theta = \frac{b}{c}$

$\cos\theta = a/c$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}$.




Measure $\theta$ by arc length, ie the circle has $2\pi$ radians

Facts: $\sin\theta$, $\cos\theta$ have period $2\pi$

$\tan\theta$ " " $\pi$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, ...

Also sin, cos, tan $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \ldots$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\left(\frac{5\pi}{2}\right) = \cos\left(2\pi + \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

See CLP Appendix for "expected" background on exp, log, trigonometry.

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<u>Facts:</u> $(\sin\theta)' = \cos\theta$, $(\cos\theta)' = -\sin\theta$.

$(\tan\theta)' = \frac{1}{\cos^2\theta} = 1 + \tan^2\theta$

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$\left[\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right]$

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## 2. Trigonometric functions

(3) (Special values) What is $\sin\frac{\pi}{3}$? What is $\cos\frac{5\pi}{2}$?

(4) Derivatives of trig functions

(a) Interpret $\lim_{h\to 0}\frac{\sin h}{h}$ as a derivative and find its value.

to get

$$\lim_{h\to 0}\frac{\sin h}{h} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ choose $f(x)=\sin x$, $a=0$

ok since $f(a)=\sin 0=0$. Then $\lim_{h\to 0}\frac{\sin h}{h} = f'(0)=\cos 0 = 1$.

(if we recognize a limit as a derivative, can compute it using differentiation facts)

(b) Differentiate $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\frac{d(\tan\theta)}{d\theta} = \frac{(\sin\theta)'\cos\theta - \sin\theta(\cos\theta)'}{\cos^2\theta} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} =$$

$$= \frac{1}{\cos^2\theta} = 1+\tan^2\theta$$

$\sin^2\theta + \cos^2\theta = 1$