Last time: transcendence basis is a maximal alg. indep. subset of a field.

$L/k$ extension, want $\mathcal{E}\subseteq\text{indep }L/k$.

Write $L = K(\mathcal{E}) > K$ where $K(\mathcal{E})/k$ purely transcendental, $L/K(\mathcal{E})$ algebraic.

Wanted to show $\#\mathcal{E}$ independent of choice of basis. (called this the transcendence degree, wrote $\text{tr.deg}_k(L) = |\mathcal{E}|$.

Key step: exchange lemma: if $A, B$ alg. indep. sets, can move elements from $B$ to $A$ if we remove corresponding elements from $A$.

Key step for this: if a polynomial $f \in k[t]$ "involves" variable $t$, then same true of any multiple.

(really had $f \in k[t][x]$)
If we write \( f = \sum a_k x^k \), \( a \in K[\mathbb{T}] \).

We saw: if \( a_0 \) involves \( t \), same tree for \( \text{const coef} \) of any multiple. (\text{const coef} \) of \( f \) has fewer variables than \( f \).

In general, let \( f \) be smallest \( s.t. t \) occurs in \( a_j \), suppose \( f \cdot g \) does not contain \( t \), \( g \in K[\mathbb{T}][x] \).

Then \( g = \sum_{l=0}^{\infty} b_l x^l \), then \( b_0 a_0 = \text{const coef of} \ f g \) does not contain \( t \), so \( b_0 \) does not contain \( t \).

**Polynomial**: \( f \in K[\mathbb{T}] \) means \( f \) is a finite sum \( f = \sum a_i t_i^{\alpha_i} \) where \( a_i \in K \), \( \alpha_i \in \mathbb{N}_0 \) of finite support.

\[ a_2 \cdot (t_1, t_2) + a_3 \cdot t_1 t_2 + \cdots \]

Say \( f \) contains \( t \), if for some \( i \), \( a_i \neq 0 \) and \( \sigma_f(t) > 0 \).
Write \( f \) as a polynomial in \( K[T,t,s][t_i] \) so \( f = \sum_{k \geq 0} a_k t_i^k, \; a_k \in K[T,t,s][t_i] \).

Multiply by some \( g \in K[T] = K[T,t,s][t_i] \) to add ( \( K[T,t,s][t_i] \) is an integral domain) so \( fg \) has positive degree in \( t_i \).

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dave \( O(T,e) \) want to put \( e+\pi \) in there. Min poly of \( e+\pi \) is \( x-e+\pi \).

(can treat \( e+\pi \) as "variables")

Since no expression in \( e+\pi \) is zero.

want to show \( \{T, T_2, \ldots \} \) are independent.

\[ \text{If min poly of } T \text{ over } O(T) \text{ is a multiple of the min poly of } T \text{ over } O(T,e) \]

but then \( e \) would occur in \( h \), contradicting \( h \in O(T, e+\pi) \).
If we want to add \( b \) to \( A \), might remove a \( a_i \) of \( \text{Gal}(A_1, \{a_i\}) \) is not indep, \( b \) depends on \( A_1, \{a_i\} \).

So has min poly \( h \in K[[A_1, \{a_i\}]] [x] \).

Then \( h \) is a multiple of min poly \( f \) of \( b \) over \( K(A) \), \( \exists f \in K(A)[x] \). But \( f \) is a multiple of \( f \) : \( h(b) = 0 \)

So if we choose a \( a_i \) to occur in \( f \), it would occur in \( h \) too: contradiction. (Some a must occur in \( f \) if \( b \) in transcendental over \( K \))

\[ \text{Infinite algebraic extensions} \]

For finite extensions we defined:
- \( LK \) be algebraic
- Recall \( LK \) separable of \( K \), min poly of \( \alpha \) has distinct roots in its splitting field,
- \( LK \) normal if \( \text{Ker} \), min poly of \( \alpha \) splits.
Saw if \( L/K \) finite then

(1) \( L/K \) separable \( \Rightarrow \) \( L/K \) separable by separable elements

\( \Leftrightarrow \) \( L \) having \( [L : K] \) embeddings into a normal closure.

(2) \( L/K \) normal \( \Rightarrow \) \( L/K \) is a splitting field if normal closures exist.

Next: (1) \( L/K \) separable \( \Rightarrow \) each \( L/K \) separable elements still true for all extensions

(2) normal closures still exist.

(3) if \( N/K \) normal, \( L/K \) algebraic,
then \( \text{Hom}_K (L, N) \) is an orbit of \( \text{Aut}_K (N) \).

Examples of \( \infty \) extensions:

\( \mathbb{Q} : \mathbb{Q} \), \( \mathbb{F}_p : \mathbb{F}_p \)
\[ \mathbb{Q}(\sqrt{p}) = \mathbb{Q}(\sqrt{p}, \sqrt{p^3}, \sqrt{p^5}, \ldots) \]
\[ \mathbb{Q}(\sqrt[3]{p}) = \mathbb{Q}(\sqrt[3]{p}, \sqrt[3]{p^3}, \ldots) \]  
Fact: this is the maximal abelian extension of \( \mathbb{Q} \), i.e.

\[ \text{Gal}(\mathbb{Q}(\sqrt[3]{p}) : \mathbb{Q}) = \text{Gal}(\mathbb{Q} : \mathbb{Q}) \]  

Difficulty: what do we mean by \( p \)?

Fact: \( \text{Gal}(L/K) \) (if \( L/K \) is Galois) comes with a topology (notion of open & closed subsets) so that action on \( L \) is continuous

\( \Rightarrow \) Stabilizers of subfields are closed.

(Condition \( f(x) = x \) is closed in \( f \).)

Galois correspondence:

\[ \{ \text{closed subgroups} \} \]  
\[ \{ \text{closed subfields} \} \]  
\[ \subset \]  
\[ \text{Gal}(L/K) \]  
\[ \subset \]  
\[ \text{Gal}(L/K) \]

Example: \( \text{Gal}(F_p^p : F_p) \) contains Frobenius

\[ \sigma(x) = x^p. \]

\[ \text{Then } <\sigma> = \gamma \text{ is not all the} \]
Galois group. But, it's dense: the closure is the stabilizer of \( F_2 \), i.e. the whole group

\[
\overline{\text{Savry}}: \text{Gal}(\mathbb{Q}(\sqrt{n}), \mathbb{Q}) \rightarrow (\mathbb{Z}/n\mathbb{Z})^{	imes}
\]

\[
\theta(2n), \quad \theta(2n), \quad \theta(2n), \quad \theta(2n)
\]

Let \( G : \text{Gal}(\mathbb{Q}(\mu_n), \mathbb{Q}) \)

If we believe Galois correspondence, have quotient maps \( G \rightarrow \text{Gal}(\mathbb{Q}(\sqrt{n}), \mathbb{Q}) \) for all \( n \).