Math 501, lecture 2: 14/3/2020

Admin: Problem session Fri 13:30 - 14:30
Office hours: Tuesdays 21:00 - 23:00

Last time: Normal series \( G = G_0 \triangleleft G_1 \triangleleft G_2 \cdots \triangleleft G_k \triangleleft \), \( k \geq 2 \)?

Solvable gp: choose series with \( G_i / G_{i+1} \) abelian

HW: \( G = GL_2(\mathbb{F}_3) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \), \( B = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \), \( B \triangleleft G \), \( G / B \) solvable.

Example: \( |G| = 12 \Rightarrow G \) solvable.

Example: Every gp of order \( p^2 q \) is solvable (\( p, q \) distinct primes)

PF: Recall Sylow thm: (1) There exist subgp of order \( p^2 \)

(2) \( p \cdot \text{Sylow subgp} ) \) all conjugate (3) \( \# \equiv 1 \mod p \)

- If there is a unique \( p \cdot \text{Sylow subgp} P \), it's normal

- \( P \) is abelian, \( G / P \cong C_{q} \) is abelian, so we're done.

- Else, \( \text{Stab}_G(P) = N_G(P) \triangleleft G \), but \( [G : P] = q \).

So \( N_G(P) = P \), so \( \Phi_P : \{ P, \ldots, P \} \) (orbit-stabilizer theorem).

\( \Rightarrow q \equiv 1 \mod p \), \( q \geq p+1 \)

- If there is a unique (normal) \( q \cdot \text{Sylow subgp} Q \)}
then $G$ is abelian (order $p^2$), $G/Q$ abelian (order $p^2$), so $G$ is solvable.

Otherwise, # of $q$-Sylow subgroups divides $p^2$, $q = 1 | G$ so this $g = p^2$ ($q = p^1$) so # $q$-Sylow subgroups is $p^2$.

$\Rightarrow p^2 = 1 (q) \Rightarrow q$ divides either $p - 1, p + 1$, but $q > p$, so can't divide $p - 1$. So $q = p^1, q/p^1$, so $q = p^1$.

So $p = 2, q = 3$, $\# G = 2^2 \cdot 3 = 12$, $G$ is solvable.

Hence do same for $\# G = p^2 q^2$.

Then (Frobenius): if $\# G = p^2 q^2$ then $G$ is solvable.

Then (Feit-Thompson): every $G$ of odd order is solvable.

Proof: let $G$ be a gp, $H, N \leq G$, $N \triangleleft G$. Then:

1) If $G$ is solvable, so are $H, G/N$.
2) If $N, G/N$ are solvable, so is $G$.

Proof: let $1 \leq \mathcal{Z}_i \leq G_i$ be a normal series in $G$ with abelian quotients. The $\mathcal{Z}_i \cap H \mathcal{Z}_i$ is a normal series in $H$ if $g \in H, \Rightarrow g \mathcal{Z}_i \cap H = \mathcal{Z}_i$. Since $g \in G_i$, $h \in H$, $\Rightarrow g \mathcal{Z}_i \cap H = \mathcal{Z}_i$. Since $g \in H$, so $g \mathcal{Z}_i \cap H = H \mathcal{Z}_i$. $G_i$.

Also, $H_i / H \mathcal{Z}_i = H \mathcal{Z}_i G / H \mathcal{Z}_i \Rightarrow G_i / H \mathcal{Z}_i$.

Since $(H \mathcal{Z}_i \cap G_i) \rightarrow G_i / H \mathcal{Z}_i \rightarrow H \mathcal{Z}_i \cap G_i = H \mathcal{Z}_i$ so $H_i / H \mathcal{Z}_i$ is abelian.
Similarly, let \( g : G \to G/N \) be the quotient map, set
\[ N_i = g(G_i), \quad N_0 = g(G_0) = g(G) = G/N \]
\[ G_i/N_i \]
\[ N_k = g(\langle e \rangle) = \langle e \rangle \subset G/N \]

Now \( G_i \) normalizes \( G_{i+1} \) and \( N_i \), so it normalizes \( G_{i+1}/N_i \), hence also \( G_{i+1}/N_i = N_{i+1} \), so \( N_i \) normalizes \( N_{i+1} \).

Finally, the map \( G_i/N_i \to N_i/N_{i+1} \) has kernel containing \( G_{i+1} \). So \( N_i/N_{i+1} \) is a quotient of \( G_i/G_{i+1} \).

So \( N_i/N_{i+1} \) is abelian.

(2) Let \( N \subseteq G/N \) be soluble, let \( \{ N_i \} \) be a normal series in \( N \), with abelian quotients.

Let \( \{ H_j \} \) be a normal series in \( G/N \), with abelian quotients. Let \( \{ M_j \} \) be \( M_j = g^{-1}(\langle H_j \rangle) \). Get:
\[ G = N_0 > N_1 > N_2 > \cdots > N_k = N = N_0 > N_1 > \cdots > N_k = Z_G \]

Now \( N_{i+1} \subset N_i \), with abelian quotients. (By choice of \( N_i \))

And \( N_{j+1} \subset N_j \), with \( N_i/N_{i+1} \cong N_i/H_{i+j} \) (correspondence theorem).

Let \( W = N_i \cap N_{i+1} \) (as \( W \) is a subgroup of \( N_i \)).

Prove some result using the derived series (FTS problem 7)

Aside: Nilpotent groups

Extension of facts: \( G \times N \), \( N \) normal, \( G \) assembled from
\( G/N, N \) (extension of \( G/N \) by \( N \))

**Extension is abelian if** \( N \) is abelian

**Central** if \( N \) is central in \( G \) (\( N \leq Z(G) \)).

So: \( G \) is soluble if it can be obtained from \( \mathbb{Z} \) by a finite sequence of abelian extensions.

Def: \( G \) is nilpotent if it can be obtained from \( \mathbb{Z} \) by a finite sequence of central extensions.

\( \Rightarrow \) \( G \) has a series of subgroups \( G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_s = \{e\} \)

\( \phi \): \( G_i \to G_i/G_{i+1} \subseteq Z(G/G_{i+1}) \) for all \( i \).

**Example:** \( B = \mathbb{Z}(2, 3) \) \( \supseteq N = \mathbb{Z}(2, 3) \subseteq \text{nilpotent} \)

Def: \( Z^i(G) \) as follows: \( Z^0(G) = \{e\} \), \( Z^{i+1}(G) = Z(G/Z^i(G)) \)

\( Z^1(G) = Z(G) \), ...

Clearly, all \( Z^i(G) \) characteristic, hence normal.

Ex: if \( G_i \) as above then \( G_i \in Z^i(G) \)

\( Z^i(G) = \{ g \in G | g \text{ commutes with } G \mod Z^{i-1}(G) \}

\( G_{i-1}(G) = \{ g \in G | g \text{ commutes with } G \mod G_{i-1} \}

\Rightarrow \) if \( G \) is nilpotent then \( Z^s(G) = G \) for some \( s \) (converse immediate)

Also: \( [x, y] = xyx^{-1}y^{-1} \) if \( A, B \in G \) set \( \{A, B\} = \langle \{x, y\} \rangle \)
Set \( \delta_i(G) = G_i \), \( \delta_i(G) = \{ G_i, G_{i-1}(G) \} \).

All characteristic, hence normal.

Let \( G_i \) as above, \( G_i(G) \subseteq G_i \). Since \( G_i/G_{i-1} \leq (G_{i-1}/G_{i-2}) \)

\( \Rightarrow \) \( G_i(G), G_i \subseteq G_i \)

so \( G_i(G) \subseteq G_i \).

\( \Rightarrow \) if \( G \) is nilpotent, \( \delta_i(G) = \{ \} \) for some \( i \).

(Converse)

(Ascending/Descending Central Series)

Ex: \( G/\delta(G) \approx G/\{0 \}, \delta(G) \) is abelian.

Theorem: \( G \) any gp (not nec. nilpotent), \( X \in G \).

Suppose image of \( X \) generates \( G/\delta_i(G) \approx G/\delta_i(G) \).

Then the image of \( X \) generates \( G/\delta_i(G) \) for all \( i \).

Cor: \( G \) is nilpotent, \( X \) generates \( G \).

Pf: Let \( H = \langle X \rangle \). Then by induction, \( H \) surjects on \( G/\delta_i(G) \). Want to show: \( H \) surjects on \( G/\delta_i(G) \).

Let \( g \in G \). Then have \( h, i = f_i \). \( g = h_i \).

Want to show: \( h_i = f_i \).

Why we can divide by \( \delta_i(G) \), i.e. assume \( \delta_i(G) \).

Want to show: \( H \) surjects on \( G/\delta_i(G) \).

Enough to show: \( H \) contains \( \delta_i(G) \).
$\phi_1(G) = [G, \pi_{i-1}(G)]$ want to show $H$ contains this
by $\phi_i(G)$ is central in $G/\pi_{i+1}(G)$ then
image $[g, h]$ in $G/\pi_{i+1}(G)$ only depends on $g, h$ mod $\pi_i(G)$

\[ [g, h] = ghg^{-1}h^{-1} \]

\[ [g, h] = gh(\pi_{i+1}(G)) = gh^{-1} \in \pi_i(G) \cap \pi_{i+1}(G) \]

$G, \pi_i$ central

$k, k' \text{ commute with } h \text{ mod } \pi_i(G)$

But $H$ generates $G/\pi_i(G)$ contains $\pi_{i-1}(G)/\pi_{i+1}(G)$
by induction, so $H$ contains $[G, \pi_{i-1}(G)] / \pi_{i+1}(G)$ so $H$ generates $G/\pi_{i+1}(G)$

$[g, h] \in \phi_1(G)$

$G \times \pi_{i-1}(G) \rightarrow \phi_1(G)$

gives map $(G/\pi_{i+1}(G)) \times \pi_{i-1}(G)/\pi_{i+1}(G) \rightarrow \phi_1(G)/\pi_{i+1}(G)$

let $\pi_i(G)/\pi_{i+1}(G)$ so map really is

central $G/\pi_i(G) \times \pi_{i-1}(G)/\pi_{i+1}(G) \rightarrow \phi_1(G)/\pi_{i+1}(G)$

so $H \times$ generates $G/\phi_1(G)$ it generates every element in $\phi_1(G)/\pi_{i+1}(G)$ so $H$ generates $G/\phi_1(G)$

Concretely $\pi_1 = \langle 1, \phi_x \rangle \setminus \{ 0 \}$ $\pi_{0, \phi_x} = \langle 1, 0 \rangle$
\[ N^2 / [N^2 , N^2 ] = \bigoplus_{i} \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^2 : (2) \mapsto \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \]

\[ \Rightarrow \mathbb{N}_2 (\mathbb{Z}) \text{ generated by} \quad \left( \begin{array}{ccc} 1 \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{ccc} 0 \\ 1 \\ 0 \end{array} \right) \]

Remarks:
1) Every \( p \)-gp. is nilpotent:
\[ 2 (G) \neq \{ e \}, \text{ G/H(G) nilp by induction} \]

(2) Thm: A finite gp. \( G \) is nilpotent if for each \( p \) or \( G \) has a uniquely-solvable Sylow
\[ \Rightarrow G \cong \text{direct product} \]
\( p \)-gps.