Lior Silberman’s Math 501: Problem Set 2 (due 25/9/2020)

Some polynomial algebra

1. Let \( R \) be a ring, \( P \in R[x] \).
   (a) Show that \((x - y)\) divides \((x^n - y^n)\) in \( \mathbb{Z}[x,y] \), and conclude that if \( a \in R \) is such that \( P(a) = 0 \) then \((x - a)\) divides \( P(x) \) in \( R[x] \).
   (b) Suppose now that \( R \) is an integral domain and that \( \{a_i\}^k_{i=1} \subset R \) are distinct zeroes of \( P \). Show that \( \prod_{i=1}^k (x - a_i) | P \) in \( R[x] \). Give a counterexample when \( R \) has zero-divisors.

2. (The Vandermonde determinant) Let \( V_n(x_1, \ldots, x_n) \in M_n(\mathbb{Z}[x_1, \ldots, x_n]) \) be the \( n \times n \) Vandermonde matrix \((V_n)_{ij} = x_i^{j-1}\). Let \( V_n(x) = \det(V_n(x)) \in \mathbb{Z}[x_1, \ldots, x_n] \) (in other words, the entries of \( V_n \) come from the ring of polynomials in \( n \) variables, and hence its determinant is also in this ring).
   (*a) Show that there exists \( c_n \in \mathbb{Z} \) so that \( V_n(x) = c_n \prod_{i>j}(x_i - x_j) \).
   Hint: You know \( n - 1 \) zeroes of \( V_n \), thought of as an element of \((\mathbb{Z}[x_1, \ldots, x_{n-1}])[x_n] \).
   (b) Setting \( x_n = 0 \) show that \( c_n = c_{n-1} \), hence that \( c_n = 1 \) for all \( n \).
   SUPP (Lagrange interpolation) Let \( F \) be a field. Show that for any \( \{(x_i, y_i)\}^n_{i=1} \subset F^2 \) with the \( x_i \) distinct there is a unique polynomial \( p \in F[x] \) of degree at most \( n - 1 \) such that \( p(x_i) = y_i \).

Irreducible polynomials and zeroes

3. Let \( f \in \mathbb{Z}[x] \) be non-zero and let \( \frac{a}{b} \in \mathbb{Q} \) be a zero of \( f \) with \((a,b) = 1\). Show that constant coefficient of \( f \) is divisible by \( a \) and that the leading coefficient is divisible by \( b \). Conclude that if \( f \) is monic then any rational zero of \( f \) is in fact an integer.

4. Decide while the following polynomials are irreducible
   (a) \( t^4 + 1 \) over \( \mathbb{R} \).
   (b) \( t^4 + 1 \) over \( \mathbb{Q} \).
   (c) \( t^3 - 7t^2 + 3t + 3 \) over \( \mathbb{Q} \).

5. Show that \( t^4 + 15t^4 + 7 \) is reducible in \( \mathbb{Z}/3\mathbb{Z} \) but irreducible in \( \mathbb{Z}/5\mathbb{Z} \). Conclude that it is irreducible in \( \mathbb{Q}[x] \).

Derivations and differential rings

* Differential rings will be a source of some advanced examples in this course covered only in problem sets. It’s ok to skip this material.

* In a ring \( R \) for any \( n \in \mathbb{Z}_{\geq 0} \) we identify \( n = \underbrace{1_R + \cdots + 1_R}_n \) (and similarly for \(-n\)).

6. Let \( R \) be a ring, and let \( S \) be an \( R \)-algebra (a ring with a compatible structure as an \( R \)-module). A \textit{derivation} on \( S \) is an \( R \)-linear map \( \partial : S \to S \) such that \( \partial(fg) = \partial f \cdot g + f \cdot \partial g \) for all \( f, g \in S \). A \textit{differential \( R \)-algebra} is a pair \((S, \partial)\) with \( S, \partial \) as above.
   (a) Call \( f \in S \) \textit{constant} if \( \partial f = 0 \). Show that the set of constants is a subring containing the image of \( R \) in \( S \).
   (b) Show that \( \partial(f^n) = n f^{n-1} \partial f \) for positive \( n \), and also for negative \( n \) if \( f \) is invertible (hint: apply \( \partial \) to \( f^n \cdot f^{-n} = 1 \)). Conclude that if \( S \) is a field then the set of constants is a subfield, the \textit{field of constants}.
   (c) Show that for any open interval \( I \subset \mathbb{R} \), \( \frac{d}{dx} \) is a derivation on the \( \mathbb{R} \)-algebra \( C^\infty(I) \).
   (d) Show that for any ring \( R \), \( \partial(\sum_{n=0}^{\infty} a_n x^n) = \sum_{n=1}^{\infty} n a_n x^{n-1} \) defines a derivation on \( R[[x]] \), the \textit{formal derivative}. Show that if \( R \) is an integral domain then the constants of \( \partial \) are exactly the constant series \( a_0 + \sum_{n=1}^{\infty} 0 x^n \).
   SUPP Let \( \partial_1, \partial_2 \) be derivations on \( S \) and let \( \alpha \in R \). Show that \( f \mapsto \alpha \partial_1 f \) and \( f \mapsto \partial_1 \partial_2 f - \partial_2 \partial_1 f \) are derivations. This makes the set of derivations into a \textit{Lie algebra} over \( R \).
Supplementary Problems I: Review of ideals

Fix a ring $R$.

A. (Working with ideals)
(a) Let $I$ be a set of ideals in $R$. Show that $\bigcap I$ is an ideal.
(b) Given a non-empty $S \subset R$ show that $(S) \triangleq \bigcap \{ I \mid S \subset I \triangleleft R \}$ is the smallest ideal of $R$ containing $S$.
(c) Show that $(S) = \{ \sum_{i=1}^{n} r_i s_i \mid n \geq 0, r_i \in R, s_i \in S \}$.
(d) Let $a \in R^\times$. Show that $a$ is not contained in any proper ideal.
   \textit{Hint:} Show that $a \in I$ implies $1 \in I$.

B. (Prime and maximal ideals) Call $I \triangleleft R$ \textit{prime} if whenever $a, b \in R$ satisfy $ab \in I$, we have $a \in I$ or $b \in I$.
Call $I$ \textit{maximal} if it is not contained in any proper ideal of $R$.
(a) Show that $R$ is an integral domain iff $(0) = \{0\} \triangleleft R$ is prime.
(b) Show that $I \triangleleft R$ is prime iff $R/I$ is an integral domain.
(c) Show that $R$ is a field iff $(0)$ is its unique ideal (equivalently, a maximal ideal).
(d) Use the correspondence theorem to show that $I$ is maximal iff $R/I$ is a field.
(e) Show that every maximal ideal is prime.
   \textit{Hint:} Every field is an integral domain.