Lior Silberman's Math 501: Problem Set 1 (due 18/9/2020)

Practice problems, any sub-parts marked "OPT" (optional) and supplementary problems are not for submission. RMK are remarks. Starred problems are more difficult.

Review of group theory

1. (Cyclic groups)
   (a) Which groups have no non-trivial proper subgroups?
   (b) Show that the infinite cyclic group $\mathbb{Z}$ is the unique group which has non-trivial proper subgroups and is isomorphic to all of them.

2. (Groups with many involutions) Let G be a finite group, and let $I = \left\{g \in G \mid g^2 = e\right\} \setminus \{e\}$ be its subset of *involutions* (e is the identity element of G).
   (a) Show that G is abelian if it has *exponent* 2, that is if $G = I \cup \{e\}$.
   (**b) Show that G is abelian if $|I| \geq \frac{3}{4}|G|$.

3. Fix a set X. The *support* of a permutation $\sigma \in S_X$ is the set $\mathrm{supp}(\sigma) = \{x \in X \mid \sigma(x) \neq x\}$.
   (a) Let $F_X \subset S_X$ be the set of permutations of finite support. Show that F_X is a normal subgroup.
   (b) Show that F_X is generated by transpositions, and that there is a homomorphism $\mathrm{sgn}\colon F_X \to \{\pm 1\}$ taking the value -1 on all transpositions. Write A_X for its kernel.
   (*c) Suppose X is infinite. Show that A_X is a simple group. You may use the fact that A_n are simple for $n \geq 5$.

Composition series and solvable groups

4. Find a group which has no composition series.

5. Show that every group of order p^2q^2 is solvable.

6. Let R be a ring. Let $G = \mathrm{GL}_n(R)$ be the group of invertible $n \times n$ matrices with entries in R, let $B < G$ be the subgroup of upper-triangular matrices, $N < B$ the subgroup of matrices with 1s on the diagonal. Next, for $0 \leq j \leq n-1$ write N_j for the matrices with 1s on the main diagonal and 0s on the next j diagonals. When $n = 4$ we have: $N = N_0 = \left\{\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}\right\}$, $N_1 = \left\{\begin{pmatrix} 1 & 0 & * & * \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right\}$, $N_2 = \left\{\begin{pmatrix} 1 & 0 & 0 & * \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right\}$.
   (a) Show that $N \lhd B$ and that $B/N \simeq (R^\times)^n$ (direct product of n copies).
   (b) For each $0 \leq j < n-1$, $N_{j+1} \lhd N_j$ and $N_j/N_{j+1} \simeq R^{n-j-1}$ (direct products of copies of the additive group of R).
   (c) Conclude that B is solvable.
   RMK When F is a field (and even more generally) B is a maximal solvable subgroup of G.

7. (The derived series) Fix a group G and recall that its *derived series* is defined by $G^{(0)} = G$ and $G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right] = \left(G^{(i)}\right)'$.
   (a) Suppose $G^{(k)} = \{e\}$ for some k. Show that G is solvable.
   (b) Suppose that $G_k \lhd G_{k-1} \lhd \cdots \lhd G_0 = G$ (note that we don't require $G_k = \{e\}$) and that the quotients G_i/G_{i+1} are all abelian. Show that $G_i \supset G^{(i)}$ for all $0 \leq i \leq k$.
   (c) Conclude that G is solvable iff $G^{(k)} = \{e\}$ for some k.

SUPP A subgroup $H < G$ is *characteristic* if for any automorphism $\alpha \in \mathrm{Aut}(G)$ we have $\alpha(H) = H$. Write H chr G.

(a) Show that characteristic subgroups are normal.
(b) Suppose K chr H chr G. Show K chr G.
(c) Show that the center $Z(G)$ and the $G^{(i)}$ are characteristic subgroups.
(d) Suppose K chr $H \lhd G$. Show that $K \lhd G$.

Supplementary Problems I: More examples of groups

Definition. Let F be a field, V an F-vectorspace. An *affine combination* in V is a sum $\sum_{i=1}^{n} t_i \underline{v}_i$ where $t_i \in F$, $\underline{v}_i \in V$ and $\sum_{i=1}^{n} t_i = 1$. If V, W are vector spaces then a map $f \colon V \to W$ is called an *affine map* if for every affine combination in V we have

$$f\left(\sum_{i=1}^{n} t_i \underline{v}_i\right) = \sum_{i=1}^{n} t_i f\left(\underline{v}_i\right) .$$

A. (The affine group) Let U, V, W be vector spaces over F, $f \colon U \to V$, $g \colon V \to W$ affine maps.

(a) Show that $g \circ f \colon U \to W$ is affine.
(b) Assume that f is bijective. Show that its set-theoretic inverse $f^{-1} \colon V \to U$ is an affine map as well.
(c) Let $\mathrm{Aff}(V)$ denote the set of invertible affine maps from V to V. Show that $\mathrm{Aff}(V)$ is a group, and that it has a natural action on V.
(d) Assume that $f(\underline{0}_U) = \underline{0}_V$. Show that f is a linear map.

B. (Elements of the affine group)

(a) Given $\underline{a} \in V$ show that $T_{\underline{a}}\underline{x} = \underline{x} + \underline{a}$ ("translation by $\underline{a}$") is an affine map.
(b) Show that the map $\underline{a} \mapsto T_{\underline{a}}$ is a group homomorphism from the additive group of V to $\mathrm{Aff}(V)$. Write $\mathbb{T}(V)$ for the image.
(c) Show that $\mathbb{T}(V)$ acts transitively on V. Show that the action is *simple*: for any $\underline{x} \in V$, $\mathrm{Stab}_{\mathbb{T}(V)}(\underline{x}) = \{T_{\underline{0}}\}$.
(d) Fixing a basepoint $\underline{0} \in V$, show that every $A \in \mathrm{Aff}(V)$ can be uniquely written in the form $A = T_{\underline{a}}B$ where $\underline{a} \in V$ and $B \in \mathrm{GL}(V)$. Conclude that $\mathrm{Aff}(V) = \mathbb{T}(V) \cdot \mathrm{GL}(V)$ setwise.
(e) Show that $\mathbb{T}(V) \cap \mathrm{GL}(V) = \{1\}$ and that $\mathbb{T}(V)$ is a normal subgroup of $\mathrm{Aff}(V)$. Show that $\mathrm{Aff}(V)$ is isomorphic to the semidirect product $\mathrm{GL}(V) \ltimes (V, +)$.

C. Let k be field, V a vector space over k of dimension n. A *maximal flag* F in V is a sequence $\{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = V$ of subspaces. Let $\mathcal{F}(V)$ denote the space of maximal flags in V.

(a) Show that the group $\mathrm{GL}(V)$ of all invertible k-linear maps $V \to V$ acts transitively on $\mathcal{F}(V)$.
(b) Let $F \in \mathcal{F}(V)$ and let $B < \mathrm{GL}(V)$ be its stabilizer. Let $N = \{b \in B \mid \forall j \geq 1 \forall \underline{v} \in F_j : b\underline{v} - \underline{v} \in F_{j-1}\}$. Show that N is a normal subgroup of B.
(c) Show that $B/N \simeq (k^{\times})^n$.
(d) Show that if $V = k^n$ and $F_i \subset k^n$ are the vectors supported on the first i coordinates (the "standard flag") then the groups B, N coincide with those of exercise 6.

D. Suppose now that k is a finite field with q elements, where $q = p^r$ for a prime p.

(a) What is $|\mathcal{F}(V)|$? *Hint*: For each one-dimensional subspace $W \subset V$ show that the set of flags containing W is in bijection with the set of flags $\mathcal{F}(V/W)$.
(b) Show that q is relatively prime to $|\mathcal{F}(V)|$. Conclude that B contains a Sylow p-subgroup of G.
(c) Show that N is a Sylow p-subgroup of B, hence of G.

Supplementary Problems II: More examples of rings

E. Let R be a ring.

DEF A *formal power series* over R is a formal expression $\sum_{n=0}^{\infty} a_n x^n$ where $a_n \in R$ (equivalently, it's just an infinite sequence $\{a_n\}_{n=0}^{\infty} \subset R$). We define addition and multiplication of power series in the obvious way. Write $R[[x]]$ for the set of power series over R in the variable x.

(a) Verify that $R[[x]]$ is a ring.

(b) Show that $f \in R[[x]]$ is invertible in $R[[x]]$ if and only if its constant coefficient a_0 is invertible in R.

DEF A *formal Laurent series* over R is a formal expression $\sum_{n=N}^{\infty} a_n x^n$ where $N \in \mathbb{Z}$and $a_n \in R$ (up to initial zeroes: $\frac{0}{x^2} + \frac{0}{x} + 1 + x^2 = 1 + x^2$). Denote the set of such series $R((x))$.

(c) Show that the set of formal Laurent series is also a ring.

(d) Show that $f \in R((x))$ is invertible if and only if its first non-zero coefficient is invertible. Conclude that if R is a field then so is $R((x))$.

F. (The topology of $R[[x]]$)

(a) Given $f \in R[[x]]$ and $N \geq 0$ let $U(f, N)$ be the set of all $g \in R[[x]]$ whose first N coefficients agree with those of f. Show that for any f, f', N, N' the intersection of $U(f, N)$ and $U(f', N')$ is either empty of equal to one of them. Conclude that $U(f, N)$ is a basis for a topology on $R[[x]]$.

(b) Show that the ring operations in $R[[x]]$ are continuous in this topology.

(c) Let $f \in R[[x]]$ have zero constant coefficient. Show that the series $1 + f + f^2 + f^3 + \cdots$ converges in $R[[x]]$ (in other words, the partial sums converge in the above topology) and that its sum is inverse to $1 - f$.

(d) Use (c) to give an alternative proof of problem A(b).

G. Let $I \subset \mathbb{R}$ be an open interval. Write $C^\infty(I)$ for the set of functions $f: I \to \mathbb{R}$ which are differentiable to all orders.

(a) A *constant-coefficient differential operator* is an expression of the form $\sum_{\alpha=0}^{n} a_\alpha \frac{d^\alpha}{dx^\alpha}$ where $a_\alpha \in \mathbb{R}$. Show that the set of all constant coefficient differential operators is a subring of $\mathrm{End}_{\mathbb{R}}(C^\infty(I))$ which is isomorphic to the polynomial ring $\mathbb{R}[x]$.

(b) A *variable-coefficient differential operator* is an expression of the form $\sum_{\alpha=0}^{n} a_\alpha(x) \frac{d^\alpha}{dx^\alpha}$ where $a_\alpha \in C^\infty(I)$. Show that the set of all variable-coefficient differential operators is a non-commutative subring of $\mathrm{End}_{\mathbb{R}}(C^\infty(I))$.

OPT Generalize these results to higher dimensions, replacing the interval I with a general open subset $\Omega \subset \mathbb{R}^n$.

RMK In PDE it is useful to think of $C^\infty(\Omega)$ as a module over the ring of differential operators. In both PDE and differential operators it is useful to think of the ring of differential operators as a module over $C^\infty(\Omega)$! (multiply the operator by a function).