Math 223, lecture 12

Today:
- On midterm questions
- Catch up: matrices & linear maps

Q. 16: If $T \in \text{Hom}_F(V, W)$, let $V$ depend on $S$, then $Tv$ depends on $T(s)$.

Proof: $v$ depends on $S \implies$ have $\{v_1, \ldots, v_n\} \subseteq S$

$\{a_1, \ldots, a_n\} \subseteq \mathbb{R}$

Let $v = \sum_{i=1}^n a_i v_i$

Applying $T$, set

$Tv = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i Tv_i$

This presents $Tv$ as a linear combo of vectors from $T(S)$, as claimed.

(Use depth, in opposite direction)
1 Linear dependence (25 points)

a. (8 points) State the definition of “In the vector space $V$, the vector $v \in V$ depends linearly on the subset $S \subset V$”

b. (5 points) Let $V, W$ be vector spaces, let $S \subset V$, and let $T : V \to W$ be a linear map. Suppose $v \in V$ depends linearly on $S$. Show that $Tv \in W$ depends linearly on $T(S) \subset W$.

c. (5 points) In each case decide (with proof) whether the given vector depends on the given set.

1. $V = \mathbb{R}^2, v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

2. $V = \mathbb{R}^3, v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \right\}$.

d. (7 points) Let $V$ be a vector space, and let $S, R \subset V$. Suppose $v \in V$ depends linearly on $S$, and suppose every vector of $S$ depends linearly on $R$. Show that $v$ depends linearly on $R$.

2 Linearity (20 points)

a. (5 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map, and suppose that $T\left( \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right) = u$ and $T\left( \begin{pmatrix} 5 \\ 4 \end{pmatrix} \right) = v$. Find an explicit vector $x \in \mathbb{R}^2$ such that $Tx = 2u - 3v$.

b. (5 points) Let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be the map $\varphi \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = 2x - 5y + 3z$. Show that $\varphi$ is a linear functional on $\mathbb{R}^3$.

c. (5 points) Show that $\begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ belong to $\text{Ker}\varphi$.

d. (5 points) Show that $\left\{ \begin{pmatrix} 5 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \right\}$ is a basis for $\text{Ker}\varphi$. 

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We need $s \neq 5.1$. Then $5x = 2.5(\frac{3}{4}) - 3.5(\frac{5}{4})$

$= 2.5(\frac{3}{1}) - 3(\frac{5}{1})$
3 Exponentials (5 points)

Recall that \( \mathbb{R}^\mathbb{R} \) is the space of real-valued functions on \( \mathbb{R} \), and let \( W \subset \mathbb{R}^\mathbb{R} \) be the subspace spanned by the set \( B = \{ e^{rx} \mid r \in \mathbb{R} \} \) of exponential functions. For a real number \( a \in \mathbb{R} \) let \( T_a : \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R} \) be the map given by \((T_a f)(x) = f(x + a)\). Show that \( T_a \) is a linear map, and that \( T_a(W) = W \).

(1) **Linearity:** For all \( f, g : \mathbb{R} \to \mathbb{R} \) and \( \alpha, \beta \in \mathbb{R} \),

\[
(T_a(\alpha f + \beta g))(x) = (\alpha f + \beta g)(x + a)
\]

By the structure of \( \mathbb{R}^\mathbb{R} \) and the definition of \( T_a \),

\[
= \alpha f(x + a) + \beta g(x + a)
\]

Using the definition of \( T_a \),

\[
= \alpha T_a f(x) + \beta T_a g(x)
\]

So \( T_a(\alpha f + \beta g) = \alpha T_a f + \beta T_a g \).

(2) **To study** \( T_a(W) = T_a(\text{span} B) \),

**Let's examine** \( T_a(B) \).

For real \( r \in \mathbb{R} \), let \( e^r \in \mathbb{R}^\mathbb{R} \) be the function \( e^r(x) = e^{rx} \). (technique: naming thing)
Then \( (T_a(e_r))(x) \)
\[
= e_r(x + a) e^{r(x + a)}
= e^{ra} e^x e^{ra} e_r(x)
\]
\[\Rightarrow T_a * e_r = e^{ra} e_r \]

So \( T_a * e_r \in \text{Span } B = W \)

So \( T_a(B) \subset W \).

Now \( \forall w \in W \) depends on \( B \),

So by 1b every \( Tw \) depends on \( T_a(B)W \)

\( \Rightarrow Tw \in W \). (\( W \) closed under linear combos)

\( \Rightarrow T(W) \in W \)

Want \( T(W) \supset W \) too, so need \( T(W) \supset B \)

on the other hand, \( T(e^{-ra} e_r) : \)
\[
= e^{-ra} T e_r = e^{-ra} e^x e^{ra} e_r = e_r
\]
\[ e \in \text{Span } B = W, \]
so \( e \in \text{Span } T(W) \) so \( B \subseteq T(W) \)
so \( \text{Span } B \subseteq T(W) \) (Span B = intersection of subspace containing B, T(W) is a subspace).

so \( W \subseteq T(W) \)

Together \( W \subseteq T(W) \) \( \text{gives equality.} \)

(Common way of proving equality)

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Continue connecting matrices & linear maps

Saw: \( T: U \rightarrow V \) is a linear map, \( \{y_j\}_{j=1}^m \subseteq U, \{x_i\}_{i=1}^n \subseteq V \) are bases, have unique scalars \( a_{ij} \) s.t.

\[ T y_j = \sum_{i=1}^n a_{ij} x_i \]
\[
\text{The } T(\sum_{j=1}^{m} x_j y_j) = \sum_{i=1}^{n} (\sum_{j=1}^{m} a_{ij} x_j) v_i \text{ i.e. coeff of } T \text{ wrt basis } \{v_i\} \text{ are given by applying the matrix } A = (a_{ij}) \text{ to the coeff } x \text{ of } y \text{ wrt } \{y_i\}.
\]

gives a map \(\text{Hom}_R(U, V) \to M_{n \times m}(R)\)
(this is linear by uniqueness of coeff.
Say matrix of \(T\) is \(A : U \to V\) \(\text{ and } S \text{ is } B : U \to V\).

What is the matrix of \(\alpha T + \beta S\)?

\[
(\alpha T + \beta S)y_j = \alpha Ty_j + \beta Sy_j.
\]

So, \(\alpha T + \beta S \in \text{Hom}(U, V) \Rightarrow V \subseteq \alpha T(y_j) + \beta S(y_j)
\]

\[
= \alpha \left( \sum a_{ij} v_i \right) + \beta \left( \sum b_{ij} v_i \right) \quad (\text{def } A, B)
\]
\[ \sum \alpha a_{ij} + \beta b_{ij} \nu_i \] (arithmetic in \( V \))

By uniqueness of the coefficients, these are the matrix coefficients of \( \alpha T + \beta S \)

Goal: this is a bijection.

Common approach: construct inverse map

\[ \text{Hom}_{\mathbb{R}}(V, V) \rightarrow M_{n \times m}(\mathbb{R}) \]

**Proof:** Know this map: it's the map assigning to \( A \in M_{n \times m}(\mathbb{R}) \) the linear map

\[ T_A \left( \sum_j x_j \nu_j \right) = \sum_i \left( \sum_j a_{ij} x_j \right) \nu_i \]

Well-defined: any vector has unique coefficients.

\( \nu_j \) (\( V \)) are linear:

If \( \nu = \sum_j x_j \nu_j \), \( \nu' = \sum_j x'_j \nu_j \)

Then \( \alpha \nu + \beta \nu' = \alpha \left( \sum_j x_j \nu_j \right) + \beta \left( \sum_j x'_j \nu_j \right) \)

\[ = \sum_j \left( \alpha x_j + \beta x'_j \right) \nu_j \]
So by uniqueness, \( x_j(\alpha y + \beta y') = \alpha x_j(y) + \beta x_j(y') \)

so \( T_A y = \sum_i \sum_j a_{ij} x_j(y) u_i \)

and \( T_A(\alpha y + \beta y') = \alpha T_A y + \beta T_A y' \) \quad \text{linearity of } x_i

\[
= \alpha \sum_i \sum_j a_{ij} x_j(y) u_i + \beta \sum_i \sum_j a_{ij} x_j(y') u_i \\
= \alpha T_A y + \beta T_A y'
\]

(aside: works in any dim too, need every column of \((a_{ij})\) to have finite many non zero entries).

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**Con:** We have an isom Hom\((U, V)\) \( \rightarrow M_m, n(C(R)) \)

**Con:** Let \( U, V \) be vector spaces, let \( B \subset U \) be a basis, and let \( f : B \rightarrow V \)
be any function. Then there is a unique linear map \( \varphi : U \to V \) st \( \varphi y = f(x) \) for \( y \in B \).

**Pf:** (1) Define \( \varphi (y) = \sum_j x_j (y) y_j \).

\[ = \sum_j y_j f(y) \in B \text{ if } \{y_j\}_{j \in J} \text{ which necessary by linearity, shows uniqueness.} \]

(we claim of uniqueness to figure out what the solution is)

repeat argument from above.

(2) Take basis of \( U \), expand \( f(y_j) \) in basis \( \{x_i\} \), get matrix of \( \varphi \).

Remarks: if \( \text{Span}(S) = U \)

then \( \varphi \) is determined by \( S \).

(\( \varphi \in \text{Hom}(U, V) \).)

But if \( S \) not a basis, not every function \( f : S \to V \) comes from/extends to
a linear map

(suppose $0 \in S$) (suppose $u_1 + u_2 = u_3$
(must have $T_0 = 0$) for $u_1, u_2, u_3 \in S$
must have $T_{u_1} + T_{u_2} = T_{u_3}$).

Look up "coherent states" for an example of understanding linear maps
("quantum observables" in terms of
a spanning set which is not a basis
("over-complete set of states")
(usual "alternative basis" is Fourier:
\[
\hat{f}(x) = \int f(x) e^{-2\pi i k x} \, dx
\]
$\mathbb{R}$)