Math 223, lecture 10

Kernel and image.

Admin: midterm info posted on course website.
45 mins. submit on Canvas.

**PS2 question:** A linear functional \( \mathcal{L} : V \to \mathbb{R} \)
is non-zero. Show there is \( v \in V \) s.t.
\[ \mathcal{L}(v) = 1 \]

**Attempt:** This is a linear equation, start with \( \mathcal{L}(v) = 1 \) try to solve for \( v \).

**Problem:** "solve equation" here says "if \( v \) is a solution then ..." but actually this equation is "fleppy" many solutions, no structure.

**Different approach:** see what we know \( \mathcal{L} \) is linear:
\[ \mathcal{L}(au + bv) = a\mathcal{L}(u) + b\mathcal{L}(v) \]
\( \mathcal{L} \) is non-zero: not always 0, so there
is some $y$ so that $\psi(u) = 0$

now by linearity $\psi(au) = a\psi(u)$

so if $a = \frac{1}{\psi(y)}$, $v = ay = \frac{1}{\psi(y)} y$

then $\psi(v) = \psi(ay) = a\psi(y) = 1$.

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**Today:** interaction of linear maps and subspaces.

Prop: Let $T: U \rightarrow V$ be a linear,

1. let $W \subseteq U$ be a subspace. Then $T(W) = \{ Tw \mid w \in W \} \subseteq V$
   is a subspace.

2. let $X \subseteq V$ be a subspace. Then $T^{-1}(X) = \{ u \in U \mid T(u) \in X \} \subseteq U$
   is a subspace.
**Pf:** (1) $\pi(W)$ subspace?

+ (i) $\pi(0) = 0$, so $0 \in \pi(W)$
+ (ii) Let $v, v' \in \pi(W)$ need to check $a\pi v + b\pi v' \in \pi(W)$

That $v, v' \in \pi(W)$ means: there are $w, w' \in W$ s.t. $\pi w = v$, $\pi w' = v'$.

Then $a\pi v + b\pi v' = a\pi w + b\pi w'$.
\[ = T(aw + bw') \in T(W) \]

because \( W \) is a subspace, \( aw + bw' \in W \).

(2) \( \tilde{T}^{-1}X \) subspace?

1. \( \tilde{T} u = 0, u \in X \), so \( u \in \tilde{T}^{-1}(0) \)

2. let \( u, u' \in \tilde{T}^{-1}X \). Want to know if \( au + bu' \in \tilde{T}^{-1}X \), i.e., if \( \tilde{T}(au + bu') \in X \)

So check: \( \tilde{T}(au + bu') = a\tilde{T}u + b\tilde{T}u' \in X \)

\( \tilde{T}u, \tilde{T}u' \in X \)

\& \( X \) is a subspace

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**Def.** The image of \( T \) is the subspace \( \text{Im}(T) = T(W) \in V \).

The kernel of \( T \) is the subspace \( \text{Ker}(T) = \{ u \in W | T_u = 0 \} = \tilde{T}^{-1}(0) \in U \).
Example: If $T = 0$, image $\text{Im}(T) = \{0\}$
kernel $\text{Ker}(T) = U$
(if we apply the zero map, we always get 0)
(1) If $\varphi : U \to \mathbb{R}$, PS3: either $\varphi = 0$
or $\text{Im}(\varphi) = \mathbb{R}$.

$\dim_{\mathbb{R}} \mathbb{R} = 1$ so only subspaces are $\mathbb{R}$, $\mathbb{R}$

Observe: if $\varphi(\bullet \cdot U) = 1$, if $w \in \text{Ker}(\varphi)$
then $\varphi(\cdot + w) = \varphi(\cdot) + \varphi(w)$
$= \varphi(\cdot) + 0 = \varphi(\cdot) = 1$

Thm: (Rank-Nullity Thm) let $\varphi : U \to V$
be linear. Then

$\dim_{\mathbb{R}} \text{Ker}(\varphi) + \dim_{\mathbb{R}} \text{Im}(\varphi) = \dim_{\mathbb{R}} U$

Example: PS3 has the case $\dim \text{Im}(T) = 1$
($T$ = functional)
Motivation: Want to solve linear equations $Tx = b$.

Image ($\Rightarrow$) does the eqn have a solution? (yes iff $b \in \mathbb{R}^n$)

Kernel ($\Rightarrow$) uniqueness of solution:
\[ \dim \ker(T) \Rightarrow \text{size of space of solutions.} \]

If we solve system
\[ \begin{align*}
   3x + 7y + 8z &= 5 \\
   2x + 3z &= 2
\end{align*} \]
\[ \dim \ker(T) = \# \text{ of parameters in the general solution.} \]

Proof: $T: U \to V$ want to show
\[ \dim U = \dim \ker(G) + \dim \mathbb{R}^n(G) \]

Need basis for $U$ of right size
\[ \dim \Rightarrow \text{size of basis} \]

Assemble basis of $U$ from bases of $\ker(G)$, $\mathbb{R}^n(G)$.
let $B \subset \ker(T)$ be a basis. By
let $C \subset \text{Im}(T)$ be a basis. C 

Enumerate: $B = \{y_i; i \in I\}$, $C = \{y_j; j \in J\}$.
for each $y_j$, choose $w_j \in U$ s.t. $T w_j = y_j$.
($w_j$ exist: $y_j \in \text{Im}(T)$)

Claim: $\{y_i; i \in I\} \cup \{w_j; j \in J\} \subset U$
are a basis.

(1) Suppose $\sum a_i y_i + \sum b_j w_j = 0$
Applying $T$ gets:
$\sum a_i T y_i + \sum b_j T w_j = 0$
each $T y_i = 0$ ($y_i \in \ker(T)$)$T w_j = y_j$ (choice of $w_j$)

So $\sum b_j y_j = 0$

but $C = \{y_j; j \in J\} \subset \text{Im}(T)$ is a basis,
hence linearly independent. So all $b_j = 0$. 
80 \sum_{i=0}^{\infty} a_i y_i = 0

But \ B = \{y_i\} \subseteq \text{Ker}(S) \text{ is a basis there,}

so all \ a_i = 0 \ too.