Lior Silberman's Math 223: Problem Set 12 (due 14/4/2021)

Practice problems

Section 6.2

Cauchy–Schwarz

SUPP Use induction on n to establish *Lagrange's identity:* for all $\underline{a},\underline{b}\in\mathbb{R}^n$:

$$\|\underline{a}\|^2\|\underline{b}\|^2-(\langle\underline{a},\underline{b}\rangle)^2=\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)-\left(\sum_{i=1}^n a_ib_i\right)^2=\sum_{1\le i<j\le n}\left(a_ib_j-a_jb_i\right)^2$$

(note that the Cauchy–Schwarz inequality for $\mathbb{R}^n$ follows immediately)

1. (a) Let $\{x_i\}_{i=1}^n\subset\mathbb{R}$ be n real numbers. Applying the CS inequality to the vectors $(x_1,\dots,x_n)$ and $(1,\dots,1)$, show that $\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2\le\frac{1}{n}\sum_{i=1}^n x_i^2$.

   RMK The quantities $\frac{1}{n}\sum_{i=1}^n x_i$, $\sqrt{\frac{1}{n}\sum_{i=1}^n x_i^2-\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2}$ are called respectively the *expectation* and *standard deviation* of the random variable that takes the values x_i with equal probability $\frac{1}{n}$.

   (**b) Let $\{x_i\}_{i=1}^n\subset\mathbb{R}$ be positive. The *Arithmetic Mean* of these numbers is the number AM $=\frac{1}{n}\sum_{i=1}^n x_i$. The *Harmonic Mean* is the number $\frac{1}{\frac{1}{n}\sum_{i=1}^n\frac{1}{x_i}}=\frac{n}{\sum_{i=1}^n\frac{1}{x_i}}$. Show the *inequalithy of the means* HM$\le$AM (with equality iff all the x_i are equal) by applying the CS inequality to suitable vectors.

Inner products and norms

2. The *trace* of a square matrix is the sum of its diagonal entries ($\mathrm{tr}A=\sum_{i=1}^n a_{ii}$).

   PRAC Show that $\mathrm{tr}\colon M_n(\mathbb{R})\to\mathbb{R}$ is a linear functional.

   (a) Show that for any two square matrices A,B we have $\mathrm{tr}(AB)=\mathrm{tr}(BA)$.

   (**b) Find three $2x2$ matrices A,B,C such that $\mathrm{tr}(ABC)\ne\mathrm{tr}(BAC)$.

   (c) Show that $\mathrm{tr}(S^{-1}AS)=\mathrm{tr}(A)$ if S is invertible.

   PRAC Show that $\langle A,B\rangle\stackrel{\mathrm{def}}{=}\mathrm{tr}\,(A^tB)$ is an inner product on $M_n(\mathbb{R})$

   DEF For $A\in M_{m,n}(\mathbb{C})$, its *Hermitian conjuate* is the matrix $A^\dagger\in M_{n,m}(\mathbb{C})$ with entries $a_{ij}^\dagger=\overline{a_{ji}}$ (complex conjuguate).

   (d) Show that $\langle A,B\rangle\stackrel{\mathrm{def}}{=}\mathrm{tr}\left(A^\dagger B\right)$ is a Hermitian product on $M_n(\mathbb{C})$.

DEFINITION. Let V be a real or complex vector space. A *norm* (="notion of length") on V is a map $\|\cdot\|:V\to\mathbb{R}_{\ge0}$such that

(1) $\|a\underline{v}\|=|a|\,\|\underline{v}\|$ (that is, $3\underline{v}$ is three times as long as $\underline{v}$)
(2) $\|\underline{u}+\underline{v}\|\le\|\underline{u}\|+\|\underline{v}\|$ ("triangle inequality")
(3) $\|\underline{v}\|=0$ iff $\underline{v}=\underline{0}$ (note that one direction follows from (1)).

3. (Examples of norms)

   (a) Show that $\|\underline{x}\|_\infty=\max_i|x_i|$ is a norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

   (b) Show that $\|f\|_\infty=\max_{a\le x\le b}|f(x)|$ is a norm on $C(a,b)$ (continuous functions on the interval $[a,b]$).

   (c) (Sobolev norm) Show that $\|f\|_{H^1}^2=\int_a^b\left(|f(x)|^2+|f'(x)|^2\right)\mathrm{d}x$ defines a norm on $C^\infty(a,b)$ (Hint: this norm is associated to an inner product)

Diagonalization

4. Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. For $\underline{u}\in V$ set $\varphi_{\underline{u}}(\underline{v})=\langle\underline{u},\underline{v}\rangle$. That $\varphi_{\underline{u}}\in V^*$ follows from the definition of the inner product.
   (a) Show that the map $V\to V^*$ given by $\underline{u}\to\varphi_{\underline{u}}$ is anti-linear, in that $\varphi_{c\underline{u}+\underline{u}'}=\bar{c}\varphi_{\underline{u}}+\varphi_{\underline{u}'}$.
   (b) We proved in class that if $\dim V=n<\infty$ then the map $\underline{u}\to\varphi_{\underline{u}}$ is a bijection $V\to V^*$. Show that its inverse map $V^*\to V$ is also anti-linear.

PRAC Check that the eigenvectors of the matrix $\begin{pmatrix}5&4&2\\4&5&2\\2&2&2\end{pmatrix}$ from PS10 are orthogonal.

5. Let $A\in M_n(\mathbb{C})$ be diagonable. Show that there exists $B\in M_n(\mathbb{C})$ such that $B^2=A$.

The Quantum Harmonic Oscillator, II

6*. Let $H=-D^2+M_{x^2}$ acting on $V=\left\{p(x)e^{-x^2/2}\mid p\in\mathbb{R}[x]\right\}$ as in PS10.
   SUPP Show that $\langle f,g\rangle=\int_{-\infty}^{+\infty}\bar{f}g\,dx$ defines an inner product on V (the difficulty is to show the integrals converge)
   SUPP Show that $i\frac{d}{dx}$ is a self-adjoint operator on V.
   (a) Show that $M_g^\dagger=M_{\bar{g}}$ for any polynomial g.
   (b) Show that $-\frac{d^2}{dx^2}$ is self-adjoint and conclude that H is self-adjoint.
   (c) Let $V_n=\left\{p(x)e^{-x^2/2}\mid p\in\mathbb{R}^{<n}[x]\right\}$. Applying the spectral theorem on V_n and V_{n+1} show that H has a unique eigenvector of the form $h_n(x)e^{-x^2/2}$ where $h_n\in\mathbb{R}[x]$ has degree exactly n.
   (d) Examining the leading coefficient of $H\left(h_ne^{-x^2/2}\right)$ show that the eigenvalue is $2n+1$.
   (**e) Show that the h_n are (up to normalization) exactly the Hermite polynomials of PS11.

Supplementary problem: Fourier series

A In this problem we use the standard inner product on $C(-\pi,\pi)$.
   (a) Show that $\left\{\frac{1}{\sqrt{2\pi}}\right\}\cup\left\{\frac{1}{\sqrt{\pi}}\cos(nx),\frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$ is an orthonormal system there.
   (b) Let a_0,a_n,b_n be the coefficient of $f(x)=2\pi|x|-x^2$ with respect to $\frac{1}{\sqrt{2n}}$, $\frac{1}{\sqrt{\pi}}\cos(nx)$, $\frac{1}{\sqrt{\pi}}\sin(nx)$. Find these.
   (c) Show that for any x, the series $\frac{1}{\sqrt{2\pi}}a_0+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\left(a_n\cos(nx)+b_n\sin(nx)\right)$ is absolutely convergent.
   FACT1 The system above is *complete*, in that the only function orthogonal to the span is the zero function. If we denote the partial sums $\left(S_Nf\right)(x)=a_0\frac{1}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{N}\left(a_n\cos(nx)+b_n\sin(nx)\right)$, this shows $S_Nf\xrightarrow[N\to\infty]{}f$ "on average" in the sense that $\|f-S_Nf\|_{L^2(-\pi,\pi)}^2=\int_{-\pi}^{\pi}\left|f(x)-\left(S_Nf\right)(x)\right|^2\mathrm{d}x\xrightarrow[N\to\infty]{}0$ (in fact, this holds for any f such that $\int_{-\pi}^{+\pi}|f(x)|^2\mathrm{d}x<\infty$).
   FACT2 For any $x\in(-\pi,\pi)$ if the sequence of real numbers $\left\{\left(S_Nf\right)(x)\right\}_{N=1}^{\infty}$ converges, and if f is continuous at x, then limit of the sequence is $f(x)$.
   (d) Conclude that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, a discovery of Euler's.