Lior Silberman's Math 223: Problem Set 6 (due 1/3/2021)

Practice problems (recommended, but do not submit)

A. Let U,V,W,X be vector spaces.
   (a) Let $A\in\mathrm{Hom}(U,V)$, $B\in\mathrm{Hom}(W,X)$. We define maps $R_A\colon \mathrm{Hom}(V,W)\to\mathrm{Hom}(U,W)$, $L_B\colon \mathrm{Hom}(V,W)\to\mathrm{Hom}(V,X)$ and $S_{A,B}\colon \mathrm{Hom}(V,W)\to\mathrm{Hom}(U,X)$ by $R_A(T)=TA$, $L_B(T)=BT$, $S_{A,B}(T)=BTA$. Show that all three maps are linear.
   (b) Suppose that $A,B\in\mathrm{Hom}(U,U)$ are invertible, with inverses A^{-1},B^{-1}. Show that AB is invertible, with inverse $B^{-1}A^{-1}$ (note the different order!)
   (c) Let $A\in\mathrm{Hom}(U,V)$, $B\in\mathrm{Hom}(V,W)$. Show that $\mathrm{Ker}A\subset\mathrm{Ker}(BA)$ and that $\mathrm{Im}(BA)\subset\mathrm{Im}(B)$.
   (d) Let $A\in\mathrm{Hom}(U,V)$, $B\in\mathrm{Hom}(V,W)$. If BA is injective then so is A. If BA is surjective then so is B.

B. Let X be a set, and let $M_g\colon\mathbb{R}^X\to\mathbb{R}^X$ be the operator of multiplication by $g\in\mathbb{R}^X$. Show that M_g is linear.

Isomorphism of vector spaces

Let U,V be two vector spaces.

C. Fix a basis $B\subset U$.
   (*a) Let $f\in\mathrm{Hom}(U,V)$ be a linear isomorphism. Show that the image $f(B)=\{f(\underline{v})\mid\underline{v}\in B\}$ is a basis of V.
   RMK It follows that is U is isomorphic to V then $\dim U=\dim V$.
   (**b) Conversely, suppose that $B'\subset V$ is a basis, and and that $g\colon B\to B'$ is a function which is $1-1$ and onto (see notations file). Show that there is an isomorphism $f\in\mathrm{Hom}(U,V)$ which agrees with g on B.
   RMK It follows that if $\dim U=\dim V$ then U is isomorphic to V.

D. Let $T\in\mathrm{Hom}(U,V)$, $S\in\mathrm{Hom}(V,U)$. Show that the following are equivalent
   (1) $ST=\mathrm{Id}_V$, $TS=\mathrm{Id}_U$.
   (2) S is invertible with inverse T.

1. Suppose that $\dim U=\dim V<\infty$. Let $A\in\mathrm{Hom}(U,V)$. Show that the following are equivalent:
   (1) A is invertible.
   (2) A is surjective.
   (3) A is injective.

Linear equations

2. (Recognition) Express the following equations as linear equations by finding appropriate spaces, linear map, and constant vector.
   (a) $$\begin{cases} 5x+7y &= 3\\ z+2x &= 1\\ 2y+x+3z &= -1\\ x+y &= 0\end{cases}.$$
   (b) (Bessel equation) $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+(x^2-\alpha^2)y=0$. Use the space $C^\infty(\mathbb{R})$ of functions on $\mathbb{R}$ which can be differentiated to all orders.
   (*c) Fixing $S,B\in\mathrm{Hom}(U,U)$ with S invertible, $SXS^{-1}=B$ for an unknown $X\in\mathrm{Hom}(U,U)$PRAC. (Show that the map you define is linear!)
   PRAC Suppose that $\dim U=n$. Using a basis for U, replace the equation of (c) with a system of n^2 equations in n^2unknowns.

Similarity of matrices.

Let U be a vector space. Write $\mathrm{End}(U)$ for $\mathrm{Hom}(U,U)$ (linear maps from U to itself). We develop here a crucial concept.

DEFINITION. We say that two transformations $A,B \in \mathrm{End}(U)$ are *similar* if there is an invertible linear map $S \in \mathrm{End}(U)$ such that $B = SAS^{-1}$.

3. (Calculations)
   PRAC Suppose that A,B are similar and $A = 0$. Show that $B = 0$.
   (a) Suppose that A,B are similar and $A = \mathrm{Id}_U$. Show that $B = \mathrm{Id}_U$.
   (b) Show that the matrices $A = \begin{pmatrix} 0 & 2 \\ 6 & -4 \end{pmatrix}$, $B = \begin{pmatrix} -33 & 15 \\ -63 & 29 \end{pmatrix}$ are similar via the similarity transformation $S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. (For a formula for S^{-1} see PS5)
4. (Meaning of similarity) Let $\mathcal{B} = \{\underline{v}_i\}_{i\in I} \subset U$ be a basis. By a practice problem C above, $\mathcal{B}' = \{S\underline{v}_i\}_{i\in I} \subset U$ is also a basis. Let $M \in M_I(\mathbb{R})$ be the matrix of A with respet to the basis $\mathcal{B}$. Show that M is also the matrix of $B = SAS^{-1}$ with respect to the basis $\mathcal{B}'$.
   RMK We'll later show that similarity as another, different meaning: similar matrices represent *the same* transformation with respect to *different* bases.
5. (Similarity is an "equivalence relation")
   (a) show that A is similar to A for all A. (Hint: choose S wisely)
   (b) Suppose that A is similar to B. Show that B is similar to A (Hint: solve $B = SAS^{-1}$ for A).
   (c) Suppose that A is similar to B, and B is simlar to C. Show that A is similar to C.

For the rest of the problem set fix A,B,S such that $B = SAS^{-1}$. Define A^n as follows: $A^0 = \mathrm{Id}_U$ and $A^{n+1} = A^n \cdot A$.

6. (Induction practice 1)
   (a) Show that $B^0 = SA^0S^{-1}$
   PRAC Show that $B^2 = SA^2S^{-1}$ and $B^3 = SA^3S^{-1}$.
   (b) Suppose that $B^n = SA^nS^{-1}$. Show that $B^{n+1} = SA^{n+1}S^{-1}$.
   The principle of mathematical induction says that (a),(b) together show that $B^n = SA^nS^{-1}$ for all n.

SUPP (Induction practice 2) For a polynomials $p(x) = \sum_{i=0}^n a_i x^i \in \mathbb{R}[x]$ and $A \in \mathrm{End}(U)$ define $p(A) = \sum_{i=0}^n a_i A^i$. We will prove that $p(B) = Sp(A)S^{-1}$.
   (a) Suppose that p is a constant polynomial. Show that $p(B) = Sp(A)S^{-1}$.
   (b) Suppose that the formula holds for polynomials of degree at most n. Show that the formula holds for polynomials of degree at most $n+1$ (hint: if p has degree at most $n+1$ you can write it as $p(x) = a_{n+1}x^{n+1} + q(x)$ where q has degree at most n).
   RMK You will need to show that $S(aT)S^{-1} = aSTS^{-1}$ for any scalar a.
   (c) Let $q(x) = \sum_{j=0}^m b_j x^j \in \mathbb{R}[x]$ be another polynomial, and let $r(x) = p(x)q(x)$ their product in $\mathbb{R}[x]$. Show that $r(A) = p(A)q(A)$.
   RMK Part (c) seems silly, but checking that things work "the way they are supposed to" is important. To understand the motivation note that we think of polynomial as *formal expressions* rather than functions – we need to make a *definition* to interpret them as functions, and then we need to verify that this definition works as expected.