Lior Silberman’s Math 412: Problem Set 6 (due 22/10/2019)

P1. (Minimal polynomials)

Let \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} \).

(a) Find the minimal polynomial of \( A \) and show that the minimal polynomial of \( B \) is \( x^2 (x - 1)^2 \).

(b) Find a \( 3 \times 3 \) matrix whose minimal polynomial is \( x^2 \).

P2. For each of \( A, B \) find its eigenvalues and the corresponding generalized eigenspaces.

Triangular matrices

P3. Let \( L \) be a lower-triangular square matrix with non-zero diagonal entries. Find a formula for its inverse.

1. Let \( U \) be an upper-triangular square matrix with non-zero diagonal entries.

(a) Give a “backward-substitution” algorithm for solving \( Ux = b \) efficiently.

(b) Explicitely use your algorithm to solve \( \begin{pmatrix} 1 & 4 & 5 \\ 2 & 6 \\ 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \).

(c) For a general upper-triangular \( U \) give a formula for \( U^{-1} \), proving in particular that \( U^{-1} \) is again upper-triangular.

RMK We’ll see that if \( A \subset M_n(F) \) is a subspace containing the identity matrix and closed under matrix multiplication, then the inverse of any matrix in \( A \) belongs to \( A \). This applies, in particular, to the set of upper-triangular matrices.

The minimal polynomial

2. Let \( D \in M_n(F) = \text{diag} (a_1, \ldots, a_n) \) be diagonal.

(a) For any polynomial \( p \in F[x] \) show that \( p(D) = \text{diag} (p(a_1), \ldots, p(a_n)) \).

(b) Show that the minimal polynomial of \( D \) is \( m_D(x) = \prod_{j=1}^r (x - a_j) \) where \( \{a_j\}_{j=1}^r \) is an enumeration of the distinct values among the \( a_i \).

(c) Show that (over any field) the matrix \( B \) from problem P1 is not similar to a diagonal matrix.

(d) Now suppose that \( U \) is an upper-triangular matrix with diagonal \( D \). Show that for any \( p \in F[x], \ p(U) \) has diagonal \( p(D) \). In particular, \( m_D|m_U \).

3. Let \( T \in \text{End}(V) \) be diagonable. Show that every generalized eigenspace is simply an eigenspace.

4. Let \( S \in \text{End}(U), \ T \in \text{End}(V) \). Let \( S \oplus T \in \text{End}(U \oplus V) \) be the “block-diagonal map”.

(a) For \( f \in F[x] \) show that \( f(S \oplus T) = f(S) \oplus f(T) \).

(b) Show that \( m_{S \oplus T} = \text{lcm}(m_S, m_T) \) (“least common multiple”: the polynomial of smallest degree which is a multiple of both).

(c) Conclude that \( \text{Spec}_F(S \oplus T) = \text{Spec}_F(S) \cup \text{Spec}_F(T) \).

RMK See also problem B below.
5. Let $K/F$ be an extension of fields, let $V$ be a finite-dimensional $F$-vector space and let $T \in \text{End}_F(V)$. Show that the minimal and characteristic polynomials of $T_K \in \text{End}_K(V_K)$ are identical with those of $T$.

**Extra credit**

6. Let $R \in \text{End}(U \oplus V)$ be “block-upper-triangular”, in that $R(U) \subset U$.
   (a) Define a “quotient linear map” $\overline{R} \in \text{End}(U \oplus V/U)$.
   (b) Let $S$ be the restriction of $R$ to $U$. Show that both $m_S, m_{\overline{R}}$ divide $m_R$.
   (c) Let $f = \text{lcm}[m_S, m_{\overline{R}}]$ and set $T = f(R)$. Show that $T(U) = \{0\}$ and that $T(V) \subset U$.
   (d) Show that $T^2 = 0$ and conclude that $f | m_R | f^2$.
   (e) Show that $\text{Spec}_F(R) = \text{Spec}_F(S) \cup \text{Spec}_F(\overline{R})$.

**Supplementary problems**

A. (Cholesky decomposition)
   (a) Let $A$ be a positive-definite square matrix. Show that $A = LL^\dagger$ for a unique lower-triangular matrix $L$ with positive entries on the diagonal.

   **DEF** For $\epsilon \in \{\pm 1\}$ define $D^\epsilon \in M_n(\mathbb{R})$ by $D^\epsilon_{ij} = \begin{cases} \epsilon & j = i + \epsilon \\ -\epsilon & j = i \\ 0 & j \neq i, i + \epsilon \end{cases}$

   (positive) discrete Laplace operator.
   (b) To $f \in C^\infty(0,1)$ associate the vector $f \in \mathbb{R}^n$ where $f(i) = f(i/n)$. Show that $\frac{1}{n}D^+f$ and $\frac{1}{n}D^-f$ are both close to $f'$ (so that both are discrete differentiation operators). Show that $\frac{1}{n^2}D^-D^+$ is an approximation to the second derivative.
   (c) Find a lower-triangular matrix $L$ such that $LL^\dagger = A$.

B. Let $T \in \text{End}(V)$. For monic irreducible $p \in F[x]$ define $V_p = \{v \in V \mid \exists k : p(T)^k v = 0\}$.
   (a) Show that $V_p$ is a $T$-invariant subspace of $V$ and that $m_{T|V_p} = p^k$ for some $k \geq 0$, with $k \geq 1$ iff $V_p \neq \{0\}$. Conclude that $p^k | m_T$.
   (b) Show that if $\{p_i\}_{i=1}^r \subset F[x]$ are distinct monic irreducibles then the sum $\bigoplus_{i=1}^r V_{p_i}$ is direct.
   (c) Let $\{p_i\}_{i=1}^r \subset F[x]$ be the prime factors of $m_T(x)$. Show that $V = \bigoplus_{i=1}^r V_{p_i}$.
   (d) Suppose that $m_T(x) = \prod_{i=1}^r p_i^{k_i}(x)$ is the prime factorization of the minimal polynomial. Show that $V_{p_i} = \text{Ker} p_i^{k_i}(T)$. 
Lior Silberman’s Math 412: Solutions to Problem Set 6

P1. (Minimal polynomials)

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ does not satisfy any linear polynomial (if $aA + b\text{Id} = 0$ then $A = -\frac{b}{a}\text{Id}$ would be scalar. However, $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 5A + 2I$ so $A^2 - 5A + 2I = 0$ and this is the minimal polynomial.

$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$ has $B^2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & 1 \end{pmatrix}$

and $(B - 1)^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$ is then easy to check that $B^2(B - 1)^2 = 0$. Thus the minimal polynomial must be a divisor of $x^2(x - 1)^2$, and by the unique factorization theorem for polynomials any such divisor divides one of $x^2(x - 1)$ and $x(x - 1)^2$. However, $B^2(B - 1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & 1 \end{pmatrix}$ has

\[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & * & * & * \\ * & * & * & * \end{pmatrix} \neq 0 \quad \text{and similarly} \quad B(B - 1)^2 \neq 0.
\]

(b) Let $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $N^2 = 0$ so the minimal polynomial is a divisor of $x^2$. The only proper divisor is $x$, and isn’t the minimal polynomial since $N \neq 0$.

P2. The eigenvalues are the roots of the minimal polynomial. For $A$ these are $\frac{5 \pm \sqrt{17}}{2}$. For $B$ these are $0, 1$. The generalized eigenspaces for $A$ are simply the eigenspaces spanned by the eigenvectors. The rest of the discussion focuses on $B$.

Let $U_0 = \text{Ker} B^2$, $U_1 = \text{Ker} (B - 1)^2$. Adding $\frac{3}{2}$ the last row to the third (assume $2$ is invertible) we see that $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 2 \\ 2 & 2 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It follows that $U_0 = \{(x, y, x + y, 0)^t\}$ (in characteristic $2$, $B^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and we get the same conclusion). Similarly, $U_1 = \{(0, 0, z, w)^t\}$. Let $V_0, V_1$ be the generalized eigenspaces. Then certainly $U_0 \subset V_0$ and $U_1 \subset V_1$. Also, $U_0, U_1$ are each $2$-dimensional and their intersection is
empty. It follows that the sum $U_0 + U_1$ is direct and 4-dimensional, that is $F^4 = U_0 \oplus U_1$. This means that $V_0 = U_0$ and $V_1 = U_1$: let $v \in V_0$, for example. Then $v = u_0 + u_1$ for $u_\lambda \in U_\lambda$. Apply $B^k$ for $k \geq 2$ such that $Bv = 0$. Then also $B^k u_0 = 0$ and so $B^k u_1 = 0$. This contradicts $B$ being invertible on $V_1$ unless $u_1 = 0$ so that $v = u_0 \in U_0$. Similarly, if $v \in V_1$ then applying $(B - I)^k$ to $v$ shows that $v \in U_1$.

2. (a) We have $D^0 = \text{Id} = \text{diag} (1, \ldots, 1) = \text{diag} (a_0, \ldots, a_0)$. Suppose that for some $k \geq 0$ we have $D^k = \text{diag} (a_1^k, \ldots, a_n^k)$. Then $D^{k+1} = DD^k = \text{diag} (a_i) \text{diag} (a_i^k) = \text{diag} (a_i^{k+1})$. Finally, let $p(x) = \sum_{k=0}^K \alpha_k x^k$. Then

$$p(D) = \sum_{k=0}^K \alpha_k D^k = \sum_{k=0}^K \alpha_k \text{diag} (a_i^k) = \sum_{k=0}^K \text{diag} \left( \sum_{k=0}^K \alpha_k a_i^k \right) = \text{diag} (p(a_i)).$$

(b) Let $p_D(x)$ be the given polynomial. Then for each $a_i$ we have $p_D(a_i) = 0$ and hence $p_D(D) = \text{diag} (0) = 0$, so the minimal polynomial divides $p_D$. On the other hand, each $a_i$ is an eigenvalue of $D$, hence a zero of $m_D$. It follows that $m_D = p_D$.

(c) Its minimal polynomial has multiple roots.

(d) Let $U, U'$ be upper-triangular. We then have $(\alpha U + U')_{ii} = \alpha U_{ii} + U'_{ii}$ and, since $U_{ij} = 0$ if $j < i$ and $U'_{ij} = 0$ if $j > i$ we have

$$(UU')_{ii} = \sum_j U_{ij} U'_{ji} = \sum_{i \leq j \leq i} U_{ij} U'_{ji} = U_{ii} U'_{ii}.$$  

Now the same induction argument as in (a) shows that $(p(U))_{ii} = p(U_{ii})$. In particular, if $p(U) = 0$ then $p(D) = 0$ and so $m_D | m_U$.

3. Let $U_\lambda \subset V$ be the eigenspaces of $T, V_\lambda$ the generalized eigenspaces.

1) Fix an eigenbasis $B \subset V$. Now suppose that $(T - \lambda)^k v = 0$ for some $v \in V$. We have $v = \sum_{i=1}^n a_i v_i$ for some $a_i \in F$ and $v_i \in B$. Suppose $T v_i = \lambda_i v_i$. Then

$$0 = (T - \lambda)^k v = \sum_{i=1}^n a_i (\lambda_i - \lambda)^k v_i.$$  

Since $B$ is a basis it follows that $a_i (\lambda_i - \lambda)^k = 0$ for each $i$, and if $\lambda_i \neq \lambda$ this means $a_i = 0$. It follows that

$$v = \sum_{i=1}^n a_i v_i \in U_\lambda.$$  

2) We have $U_\lambda \subset V_\lambda$ and at the same time $V = \bigoplus_{\lambda \in \text{Spec}_F(T)} U_\lambda$ (by assumption) and $V = \bigoplus_{\lambda \in \text{Spec}_F(T)} V_\lambda$ (theorem from class). If $V$ is finite-dimensional then we have

$$\dim_F V = \sum_{\lambda} \dim_F U_\lambda \leq \sum_{\lambda} \dim_F V_\lambda = \dim V$$  

and we must therefore have equality throughout, that is $\dim_F U_\lambda = \dim_F V_\lambda$ and $U_\lambda = V_\lambda$.

3) For each $\lambda$ let $t_\lambda : V_\lambda \to \bigoplus_{\lambda} V_\lambda$ be the standard map. Let $W \subset \bigoplus_{\lambda} V_\lambda$ be the internal direct sum of the images of the $U_\lambda$. Composing with the quotient map $\bigoplus_{\lambda} V_\lambda \to \bigoplus_{\lambda} V_\lambda / W$ gives a map

$$f_\lambda : V_\lambda \to \bigoplus_{\lambda} V_\lambda / \bigoplus_{\lambda} U_\lambda.$$  

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Note that if \( v \in U_\lambda \) then \( t_\lambda(v) \) is in \( W \), and so \( f_\lambda(v) = 0 \). It follows that \( f_\lambda \) induces a map

\[ \tilde{f}_\lambda : V_\lambda / U_\lambda \to \bigoplus \lambda V_\lambda / \bigoplus \lambda U_\lambda. \]

Finally, this family of maps induces a map

\[ \tilde{f} : \bigoplus \lambda (V_\lambda / U_\lambda) \to \bigoplus \lambda V_\lambda / \bigoplus \lambda U_\lambda. \]

This is an isomorphism: if \( v \in \bigoplus \lambda \lambda \) then \( v = \sum \lambda v_\lambda \) for some \( v_\lambda \in V_\lambda \) and then \( \tilde{f}(\sum \lambda (v_\lambda + U_\lambda)) = v + W \), and if \( \tilde{f}(\sum \lambda (v_\lambda + U_\lambda)) = 0 \) (\( \lambda \) distinct) then \( \sum \lambda v_\lambda \in W \) so each \( v_\lambda \in U_\lambda \). Now we are given that \( \bigoplus \lambda V_\lambda / \bigoplus \lambda U_\lambda \) is the zero space (both spaces are isomorphic to \( V \)) so \( \bigoplus \lambda (V_\lambda / U_\lambda) \) is zero, and hence for each \( \lambda \) \( V_\lambda / U_\lambda = \{0\} \) and \( V_\lambda = U_\lambda \).

4. Let \( S \in \text{End}(U) \), \( T \in \text{End}(V) \). Let \( S \oplus T \in \text{End}(U \oplus V) \) be the “block-diagonal map”.
(a) Let \( S_1, S_2 \in \text{End}(U), T_1, T_2 \in \text{End}(V) \) and let \( \alpha \in F \). Then

\[ [\alpha (S_1 \oplus T_1) + (S_2 \oplus T_2)](u \oplus v) = \alpha (S_1 \oplus T_1)(u \oplus v) + (S_2 \oplus T_2)(u \oplus v) = \alpha (S_1u \oplus T_1v) + (S_2u \oplus T_2v) = (\alpha S_1 + S_2)u \oplus (\alpha T_1 + T_2)v = [(\alpha S_1 + S_2) \oplus (\alpha T_1 + T_2)](u \oplus v) \]

and

\[ [(S_1 \oplus T_1)(S_2 \oplus T_2)](u \oplus v) = (S_1 \oplus T_1)(S_2u \oplus T_2v) = (S_1S_2u) \oplus (T_1T_2v) = (S_1S_2 \oplus T_1T_2)(u \oplus v). \]

Now from the second claim it follows by induction on \( k \) that \((S \oplus T)^k = S^k \oplus T^k\), and then it follows by induction on \( n \) that

\[ \sum_{k=0}^{n} \alpha_k (S \oplus T)^k = \sum_{k=0}^{n} \alpha_k (S^k \oplus T^k) = \left( \sum_{k=0}^{n} \alpha_k S^k \right) \oplus \left( \sum_{k=0}^{n} \alpha_k T^k \right). \]

(b) Let \( f = \text{lcm}(m_S, m_T) \). Then \( f(S) = 0 \) and \( f(T) = 0 \) (\( f \) is a multiple of the respective minimal polynomials), and hence \( f(S \oplus T) = f(S) \oplus f(T) = 0 \oplus 0 \), so \( f \) is divisible by the minimal polynomial of \( S \oplus T \). Conversely, we have \( m_{S \oplus T}(S \oplus T) = m_{S \oplus T}(S) \oplus m_{S \oplus T}(T) = 0 \) so \( m_{S \oplus T}(S) = 0 \) and \( m_{S \oplus T}(T) = 0 \). It follows that \( m_{S \oplus T} \) is divisible by both \( m_S \) and \( m_T \), hence by their least common multiple.

(c) Clearly if \( m_S(\lambda) = 0 \) or \( m_T(\lambda) = 0 \) then \( f(\lambda) = 0 \) (it’s a multiple). For the converse, let \( \lambda \) be a root of \( f \), but not of \( m_S \) or \( m_T \). Then \( x - \lambda \) divides \( f \) by not \( m_S \) or \( m_T \), so both \( m_S \) and \( m_T \) divide \( f(x) \), contradicting the minimality of \( f \).

5. Let \( R \in \text{End}(U \oplus V) \) be “block-upper-triangular”, in that \( R(U) \subset U \).
(a) In general, if \( T \in \text{End}(W) \) and \( Z \subset W \) is \( T \)-stable then setting \( \tilde{T}(w + Z) = Tw + Z \) gives a linear map.

(b) For any polynomial \( f \) and any \( u \in U \) we have \( f(R)u = f(S)u \), by the same induction as the in the problems above. In particular, if \( f(R) = 0 \) then \( f(S) = 0 \) and \( m_S[f] \). Similarly, for \( w \in U \oplus V \), \( f(R)(w + U) = f(R)w + U \) so that \( f(\tilde{R}) = \tilde{f}(\tilde{R}) \). In particular, if \( f(R) = 0 \) then \( f(\tilde{R}) = 0 \) and \( m_{R}[f] \).
(c) Let $f = \text{lcm}[m_S, m_R]$ and set $T = f(R)$. Since $m_S | f$ we have $f(S) = 0$. Then for $u \in U$ we have $T_u = f(S)u = 0$, so $T(U) = 0$. For the same reason, $f(\tilde{R}) = 0$, that is $\tilde{T} = 0$ which means $T(V) \subseteq U$.

(d) Let $T(V) \subset U$ and $T(U) = 0$ we have $T^2 = 0$, so $f^2(R) = 0$ and hence $m_R | f^2$. We have already seen that $f|m_R$.

(e) Since $f|m_R | f^2$, any root of $f$ is a root of $m_R$ and any root of $m_R$ is a root of $f^2$. But $f, f^2$ have the same roots.

**Supplementary problems**

A. (c) Let

$$L_{ij} = \begin{cases} \sqrt{\frac{i+1}{i}}, & j = i \\ -\sqrt{\frac{i-1}{i}}, & j = i - 1 \\ 0, & j \neq i, i - 1 \end{cases}$$

Then

$$\left(LL^T\right)_{ik} = \sum_j L_{ij}L_{kj} = L_{ii}L_{ki} + L_{i,i-1}L_{k,i-1}$$

$$= \begin{cases} 0 - \sqrt{\frac{i+1}{i}}\sqrt{\frac{i}{i+1}}, & k = i - 1 \\ \left(\sqrt{\frac{i+1}{i}}\right)^2 + \left(-\sqrt{\frac{i-1}{i}}\right)^2, & k = i \\ \sqrt{\frac{i+1}{i}}\left(-\sqrt{\frac{i-1}{i}}\right) + 0, & k = i + 1 \\ 0, & |i-k| \geq 2 \end{cases}$$

$$= \begin{cases} 2, & k = i \\ -1, & |k-i| = 1 \\ 0, & |k-i| \geq 2 \end{cases} = (-A)_{ik}.$$

B. Let $T \in \text{End}(V)$. For monic irreducible $p \in F[x]$ define $V_p = \{ \underline{v} \in V \mid \exists k : p(T)\underline{v} = 0 \}$.

(a) For a polynomial $q(x)$ we have $xq(x) = q(x)x$. Then $Tq(T) = q(T)T$. In particular, if $\underline{v} \in \text{Ker} q(T)$ then $q(T)T\underline{v} = q(T)\underline{v} = 0$ and hence $T_{\underline{v}} \in \text{Ker} q(T)$ as well. We assume that $V$ is finite-dimensional, so each $V_p$ is. In particular let $\{\underline{v}_i\}_{i=1}^n \subseteq V_p$ be a basis, and let $k$ be large enough such that $p(T)^k\underline{v}_i = 0$ for each $i$. Then $\text{Span}_F \{\underline{v}_i\}_{i=1}^n \subseteq \text{Ker} p(T)^k$. But $\text{Ker} p(T)^k \subset V_p$ by definition, so $V_p = \text{Ker} p(T)^k$. It follows that $m_T|V_p | p^k$. Since $p$ is irreducible, each divisor of $p^k$ has the form $p^{k'}$ for some $k' \leq k$. If $V_p = \{0\}$ then $m_T|V_p = 1$. Otherwise, $\text{Id}_{V_p}$ is not the zero map so $m_T|V_p \neq 1$ and $k' \geq 1$. In any case, $m_T(T | V_p) = m_T(T) | V_p = 0$ shows that $p^{k'} = m_T|V_p mt$. 

(b) Let $p, q \in F[x]$ be relatively prime (for example $p$ irreducible and not dividing $q$). We will show that $q(T)$ is invertible on $V_p$. We have $V_p = \bigcup_{k=0}^\infty \text{Ker}(p^k(T))$, so it is enough to show that $q(T)$ is invertible on each $\text{Ker}(p^k(T))$. Since $q$ is prime to $p$, it is prime to $p^k$ for each $k$. Since $F[x]$ is a PID, there are $\alpha(x), \beta(x) \in F[x]$ such that $\alpha q + \beta p^k = 1$. Then on $U = \text{Ker}(p^k(T))$ we have $1 = \alpha(T | U)q(T | U) + \beta(T | U)p^k(T | U) = \alpha(T | U)q(T | U)$,
so \( q(T) \) is indeed invertible on \( \text{Ker}(p^k(T)) \).

Now, let \( B \subset F[x] \) be a set of monic irreducibles, and let \( W = \sum_{p \in B} V_p \). We need to show the sum is direct. For this, let \( \sum_{i=1}^m \nu_i = 0 \) be a minimal dependence where \( \nu_j \in V_{p_i} \) for some distinct \( p_i \in B \). Let \( k_m \) be such that \( p_m^{k_m}(T)\nu_m = 0 \). We then have

\[
\sum_{i=1}^{m-1} p_m^{k_m}(T)\nu_i = 0.
\]

Since \( p_m^{k_m} \) is prime to \( p_i \) for \( i < m \), \( p_m^{k_m}(T) \) is invertible on \( V_{p_i} \), so \( p_m^{k_m}(T)\nu_j \neq 0 \). This contradicts the minimality of the original combination.

(c) Let \( W = \bigoplus_{i=1}^r V_{p_i} \) and suppose that \( Z = V/W \) is non-zero. Since \( W \) is \( T \)-invariant we have a quotient map \( \bar{T} \) on \( Z \). Since \( V/W \) is non-zero, we have \( 1 \neq m_T | m_T \). In particular, \( m_T \) has some irreducible factor, without loss of generality \( p_1 \). Thus let \( \nu \in V \) have non-zero image in \( Z_{p_1} \). Then \( \prod_{i=2}^r p_i^{k_i}(T) \) is invertible in \( Z_{p_1} \) so \( \prod_{i=2}^r p_i^{k_i}(T)\nu \) has non-zero image there. It follows that \( u = \prod_{i=2}^r p_i^{k_i}(T)\nu \notin W \). But \( p_1^{k_1}(T)u = m_T(T)u = 0 \) shows that \( u \in V_{p_1} \subset W \), a contradiction.

(d) Since \( m_{T|V_{p_i}} | m_T \) and has \( p_i \) as its unique irreducible divisor, we have \( m_{T|V_{p_i}} | p_i^{k_i} \). This \( p_i^{k_i}(T \mid V_{p_i}) = 0 \) and \( V_{p_i} \subset \text{Ker} p_i^{k_i}(T) \). The reverse containment holds by definition. We remark that \( k_i \) is the minimal value for which this is true: if \( p_i^{k_i-1}(T) \) vanished in \( V_{p_i} \) then \( p_i^{k_i-1}(T) \prod_{j \neq i} p_j^{k_j}(T) \) would vanish in \( \bigoplus_{j=1}^r V_{p_j} = V \), contradicting the minimality of \( m_T \).
Lior Silberman’s Math 412: Problem set 7 (due 2/11/2017)

Practice

P1. Find the characteristic and minimal polynomial of each matrix:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
5 & 0 & 0 & 0 \\
0 & 4 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
5 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

P2. Show that

\[
\begin{pmatrix}
0 & 1 \\
\alpha & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]

are similar. Generalize to higher dimensions.

The Jordan Canonical Form

1. For each of the following matrices, (i) find the characteristic polynomial and eigenvalues (over the complex numbers), (ii) find the eigenspaces and generalized eigenspaces, (iii) find a Jordan basis and the Jordan form.

\[
A = \begin{pmatrix}
1 & 2 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-2 & 1 & 0 & 1 \\
0 & 0 & 1 & 2
\end{pmatrix},
B = \begin{pmatrix}
0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix},
C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

RMK I suggest computing by hand first even if you later check your answers with a CAS.

2. Suppose the characteristic polynomial of \(T\) is \(x(x-1)^3(x-3)^4\).
   (a) What are the possible minimal polynomials?
   (b) What are the possible Jordan forms?

3. Let \(T,S \in \text{End}_F(V)\).
   (a) Suppose that \(T,S\) are similar. Show that \(m_T(x) = m_S(x)\).
   (b) Prove or disprove: if \(m_T(x) = m_S(x)\) and \(p_T(x) = p_S(x)\) then \(T,S\) are similar.

4. Let \(F\) be algebraically closed of characteristic zero. Show that every \(g \in \text{GL}_n(F)\) has a square root, in that \(g = h^2\) for some \(h \in \text{GL}_n(F)\).

5. Let \(V\) be finite-dimensional, and let \(\mathcal{A} \subset \text{End}_F(V)\) be an \(F\)-subalgebra, that is a subspace containing the identity map and closed under multiplication (composition). Suppose that \(T \in \mathcal{A}\) is invertible in \(\text{End}_F(V)\). Show that \(T^{-1} \in \mathcal{A}\).

(extra credit problem on reverse)
6. (The additive Jordan decomposition) Let $V$ be a finite-dimensional vector space, and let $T \in \text{End}_F(V)$.

DEF An additive Jordan decomposition of $T$ is an expression $T = S + N$ where $S \in \text{End}_F(V)$ is diagonal, $N \in \text{End}_F(V)$ is nilpotent, and $S,N$ commute.

(a) Suppose that $F$ is algebraically closed. Separating the Jordan form into its diagonal and off-diagonal parts, show that $T$ has an additive Jordan decomposition.

(b) Let $S,S' \in \text{End}_F(V)$ be diagonal and suppose that $S,S'$ commute. Show that $S + S'$ is diagonal.

(c) Show that a nilpotent diagonalizable linear transformation vanishes.

(d) Suppose that $T$ has two additive Jordan decompositions $T = S + N = S' + N'$. Show that $S = S'$ and $N = N'$.

**Supplementary problems: $\ell^p$ spaces**

A. For $v \in \mathbb{C}^n$ and $1 \leq p \leq \infty$ let $\|v\|_p$ be as defined in class.

(a) For $1 < p < \infty$ define $1 < q < \infty$ by $\frac{1}{q} + \frac{1}{p} = 1$ (also if $p = 1$ set $q = \infty$ and if $p = \infty$ set $q = 1$). Given $x \in \mathbb{C}$ let $y(x) = \frac{1}{|x|} |x|^{p/q}$ (set $y = 0$ if $x = 0$), and given a vector $x \in \mathbb{C}^n$ define a vector $y$ analogously.

(i) Show that $\|y\|_q = \|x\|_p^{p/q}$.

(ii) Show that for this particular choice of $y$, $|\sum_{i=1}^n x_i y_i| = \|x\|_p \|y\|_q$.

(b) Now let $u, v \in \mathbb{C}^n$ and let $1 \leq p \leq \infty$. Show that $|\sum_{i=1}^n u_i v_i| \leq \|u\|_p \|v\|_q$ (this is called Hölder’s inequality).

(c) Conclude that $\|u\|_p = \max \left\{ \|\sum_{i=1}^n u_i v_i\| \ | \|v\|_q = 1 \right\}$.

(d) Show that $\|u\|_p$ is a seminorm (hint: A(c)) and then that it is a norm.

(e) Show that $\lim_{p \to \infty} \|v\|_p = \|v\|_\infty$ (this is why the supremum norm is usually called the $L^\infty$ norm).

B. Let $X$ be a set. For $1 \leq p < \infty$ set $\ell^p(X) = \{ f : X \to \mathbb{C} \ | \ \sum_{x \in X} |f(x)|^p < \infty \}$, and also set $\ell^\infty(X) = \{ f : X \to \mathbb{C} \ | \ f \text{ bounded} \}$.

(a) Show that for $f \in \ell^p(X)$ and $g \in \ell^q(X)$ (as in A(a)) we have $fg \in \ell^1(X)$ and $|\sum_{x \in X} f(x)g(x)| \leq \|f\|_p \|g\|_q$.

(b) Show that $\ell^p(X)$ are subspaces of $\mathbb{C}^X$, and that $\|f\|_p = \left( \sum_{x \in X} |f(x)|^p \right)^{1/p}$ is a norm on $\ell^p(X)$

(c) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence. Show that for each $x \in X$, $\{f_n(x)\}_{n=1}^\infty \subset \mathbb{C}$ is a Cauchy sequence.

(d) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence and let $f(x) = \lim_{n \to \infty} f_n(x)$. Show that $f \in \ell^p(X)$.

(e) Let $\{f_n\}_{n=1}^\infty \subset \ell^p(X)$ be a Cauchy sequence. Show that it is convergent in $\ell^p(X)$.

Hint for B(d): Suppose that $\|f\|_p = \infty$. Then there is a finite set $S \subset X$ with $\left( \sum_{x \in S} |f(x)|^p \right)^{1/p} \geq \lim_{n \to \infty} \|f_n\| + 1$. 

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P1. For \( A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \), \( A - I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \exp(it) + \frac{1}{2} \exp(-it) \) so \((A - I)^2 = 0\) and the minimal polynomial is \((x - 1)^2\). The characteristic polynomial must then be \((x - 1)^4\).

For \( B = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \) we have \( V_5 = \text{Span} \{ e_1 \} , V_4 = \text{Span} \{ e_2 \} , V_2 = \text{Span} \{ e_3 , \ldots , e_6 \} \)

\((B - 5 , B - 4 , (B - 2)^2 \) vanish on the respective spaces, and they sum to \( F^6 \). The minimal polynomial is therefore \((x - 5) (x - 4) (x - 2)^2\). The characteristic polynomials on the respective spaces are \((x - 5), (x - 4), (x - 2)^4\) so on their direct sum is \((x - 5) (x - 4) (x - 2)^4\).

For \( C = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \) we have \( V_5 = \text{Span} \{ e_1 \} , V_2 = \text{Span} \{ e_2 , \ldots , e_6 \} (C - 5 , (C - 2)^3 \) vanish on the respective spaces, and they sum to \( F^6 \). The minimal polynomial is then \((x - 5) (x - 2)^3\) and the characteristic polynomial \((x - 5) (x - 2)^5\).

P2. Let \( N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). Then \( Ne_1 = 0, Ne_2 = e_1 \) and \( N (e_3 + \alpha e_2) = e_2 + \alpha e_3 \). Clearly \( \{ e_1 , e_2 , e_3 + \alpha e_2 \} \) is another basis for \( F^3 \), so \( N \) is similar to \( A = \begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). More generally, let \( A \) be a strictly upper-triangular matrix with non-zero entries right above the main diagonal. Then \( A \) is similar to the Jordan block \( N \) of the same size (ones above the main diagonal, zeroes elsewhere).

For this let \( v_n = e_n \), and for \( 0 \leq k \leq n - 1 \) set \( v_{n-k} = A^k v_n \). We show by induction on \( k \) that \( v_{n-k} \in \text{Span} \{ e_i \}_{i=1}^{n-k} \) and that the coefficient of \( e_k \) is the product \( \prod_{j=1}^{k} a_{n-j,n-j+1} \). For \( k = 0 \) the claim is evident. Suppose the claim for \( k \). Since \( A \) is strictly upper-triangular, we have \( Ae_m \in \text{Span} \{ e_i \}_{i=1}^{m-1} \). Thus if

\[
 v_{n-k} = \left( \prod_{j=1}^{k} a_{n-j,n-j+1} \right) e_{n-k} + \sum_{i=1}^{n-k-1} \alpha_i e_i
\]
then

\[ v_{n-k-1} = \left( \prod_{j=1}^{k} a_{n-j,n-j+1} \right) Ae_{n-k} + \sum_{i=1}^{n-k-1} \alpha_i Ae_i \]

\[ \in \left( \prod_{j=1}^{k} a_{n-j,n-j+1} \right) n-k-1 \sum_{i=1}^{n-k-1} a_{i,n-k} e_i + \text{Span} \{ e_j \}_{j=1}^{n-k-2} \]

\[ = \left( \prod_{j=1}^{k} a_{n-j,n-j+1} \right) a_{n-k-1,n-k} e_{n-k-1} + \text{Span} \{ e_j \}_{j=1}^{n-k-2} \]

\[ = \left( \prod_{j=1}^{k+1} a_{n-j,n-j+1} \right) e_{n-k-1} + \text{Span} \{ e_j \}_{j=1}^{n-k-2} \]

which is the claim for \( k + 1 \). Since the \( a_{i,i+1} \) are non-zero it follows that \( v_1 = \left( \prod_{i=1}^{n-1} a_{i,i+1} \right) e_1 \) is non-zero while \( Av_1 = 0 \), so \( \{ v_i \}_{i=1}^n \) is a Jordan block for \( A \) and \( A \) is similar to \( N \).

1. (a)

\[ \det(x \text{Id} - A) = \det \begin{pmatrix} x - 1 & -2 & -1 & 0 \\ 2 & x - 1 & 0 & -1 \\ 0 & 0 & x - 1 & -2 \\ 0 & 0 & 2 & x - 1 \end{pmatrix} = \det \begin{pmatrix} x - 1 & -2 \\ 2 & x - 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ = \left( \det \begin{pmatrix} x - 1 & -2 \\ 2 & x - 1 \end{pmatrix} \right)^2 = ((x - 1)^2 + 4)^2 = (x^2 - 2x + 5)^2 \]

where \( \lambda = 1 + 2i \). We find some eigenvectors:

\[ A - \lambda \text{ Id} = \begin{pmatrix} -2i & 2 & 1 & 0 \\ -2 & -2i & 0 & 1 \\ 0 & 0 & -2i & 2 \\ 0 & 0 & -2 & -2i \end{pmatrix} \]

so its eigenvectors must take the form \((x,y,z,iz)\) where \(-2ix + 2y + z = 0\), so \((x,ix - z/2, z, iz)\) that is

\[ V_\lambda \supset \text{Span}_\mathbb{C} \left\{ \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ 1 \\ i \end{pmatrix} \right\} \]

Taking complex conjugates we find

\[ V_\bar{\lambda} \supset \text{Span}_\mathbb{C} \left\{ \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ 1 \\ -i \end{pmatrix} \right\} \]

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Since the whole space is 4-dimensional, we have the eigenbasis \( \left\{ \left( \begin{array}{c} 1 \\ i \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ -1/2 \\ 1 \\ i \end{array} \right), \left( \begin{array}{c} 1 \\ -i \\ 0 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ -1/2 \\ 1 \\ -i \end{array} \right) \right\} \) so that

\[
A = SDS^{-1}
\]

where \( S = \left( \begin{array}{ccc} 1 & 0 & 1 & 0 \\ i & -1/2 & -i & -1/2 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{array} \right) \) and \( D = \text{diag} \left( 1 + 2i, 1 + 2i, 1 - 2i, 1 - 2i \right) \).

(b) \( \det \left( \text{Id} - Bx \right) = \det \left( \begin{array}{cccc} x & -1 & 1 & 0 \\ 1 & x & 0 & 1 \\ -1 & 0 & x & -1 \\ 0 & -1 & 0 & x \end{array} \right) = x \left| \begin{array}{cccc} x & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & x \end{array} \right| = x^2 \left| \begin{array}{ccc} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & x \end{array} \right| + x \left| \begin{array}{ccc} x & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & x \end{array} \right| = x^4 + x^2 + (x^2 + 1) = (x^2 + 1)^2 \). The eigenvalues are therefore \( \pm i \).

We have

\[
B - i \text{Id} = \left( \begin{array}{cccc} -i & 1 & -1 & 0 \\ 0 & -i & 0 & -1 \\ 1 & 0 & -i & 1 \\ 0 & 1 & 0 & -i \end{array} \right).
\]

Row reduction gives:

\[
B - i \text{Id} \sim \left( \begin{array}{cccc} 0 & 1 & 0 & i \\ 0 & -i & 0 & -1 \\ 1 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \sim \left( \begin{array}{cccc} 0 & 1 & 0 & i \\ 0 & 0 & 0 & -2 \\ 1 & 0 & -i & 1 \\ 0 & 2 & 0 & 0 \end{array} \right) \sim \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)
\]

so \( V_i \supset \text{Span}_{\mathbb{C}} \left\{ \left( \begin{array}{c} i \\ 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right) \right\} \) and similarly \( V_{-i} \supset \text{Span}_{\mathbb{C}} \left\{ \left( \begin{array}{c} -i \\ 0 \\ 1 \\ 0 \end{array} \right) \right\} \). Now \( (B - i)^2 = \left( \begin{array}{cccc} -2 & -2i & 2i & -2 \\ 0 & -2 & 0 & 2i \\ -2i & 2 & -2 & -2i \\ 0 & -2i & 0 & -2 \end{array} \right) \) so \( V_i \supset \text{Span} \left\{ \left( \begin{array}{c} i \\ 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} i \\ 0 \\ 0 \\ 1 \end{array} \right) \right\} \).

Similarly \( V_{-i} \supset \text{Span} \left\{ \left( \begin{array}{c} -i \\ 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} -i \\ 0 \\ 0 \\ 1 \end{array} \right) \right\} \) and since the whole space is 4-dimensional we conclude that \( \text{Span} \left\{ \left( \begin{array}{c} i \\ 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ i \\ 0 \\ 1 \end{array} \right) \right\} \) and \( \text{Span} \left\{ \left( \begin{array}{c} -i \\ 0 \\ 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ -i \\ 0 \\ 1 \end{array} \right) \right\} \) as the two Jordan block.