Math 100 – SOLUTIONS TO WORKSHEET 23
ANTIDERIVATIVES

1. Warmup: inverse operations

(1) (Multiplication)
(a) Calculate: $7 \times 8 = 56$
(b) Find (some) $a, b$ such that $ab = 15$.

(2) (Trig functions)
(a) Calculate: $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$
(b) Find all $\theta$ such that $\sin \theta = 1$.
   Solution: $\frac{\pi}{2} + 2\pi k$ or $\left\{ \frac{\pi}{2} + 2\pi k \right\}_{k \in \mathbb{Z}}$.

(3) Simple differentiation
(a) Find one $f$ such that $f'(x) = 1$.
   Solution: $f(x) = x$ works.
(b) Find all such $f$.
   Solution: $f(x) = x + C$ where $C$ is an arbitrary constant.
(c) Find the $f$ such that $f(7) = 3$.
   Solution: We need $7 + C = 3$ so $C = -4$ and hence $f(x) = x - 4$.

2. Antidifferentiation by massaging

(4) Find $f$ such that $f'(x) = 2x^3$.
   Solution: We know the derivative of $x^4$ is $4x^3$, so the derivative of $\frac{1}{2}x^4$ is $2x^3$ as desired.

(5) Find $f$ such that $f'(x) = -\frac{1}{x}$.
   Solution: We know the derivative of $\log |x|$ is $\frac{1}{x}$, so the derivative of $-\log |x|$ is $-\frac{1}{x}$.

(6) Find all $f$ such that $f'(x) = \cos 3x$.
   Solution: The derivative of $\sin x$ is $\cos x$, so the derivative of $\sin 3x$ is $3 \cos 3x$ and the derivative of $\frac{1}{3} \sin 3x$ is $\cos 3x$.

3. Combinations

(7) (Final, 2015) Find a function $f(x)$ such that $f'(x) = \sin x + \frac{2}{\sqrt{x}}$ and $f(\pi) = 0$.
   Solution: We know $(\cos x)' = -\sin x$. Also, $(x^{1/2})' = \frac{1}{2\sqrt{x}}$. The general antiderivative is therefore
   $$ f(x) = -\cos x + 4\sqrt{x} + C. $$

   To determine the constant we evaluate at $\pi$:
   $$ 0 = f(\pi) = -\cos \pi + 4\sqrt{\pi} + C = 1 + 4\sqrt{\pi} + C. $$

   We therefore have $C = -1 - 4\sqrt{\pi}$ and
   $$ f(x) = -\cos x + 4\sqrt{x} + 1 - 4\sqrt{\pi}. $$

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(8) (Final, 2016) Find the general antiderivative of \( f(x) = e^{2x^3} \).

**Solution:** Write \( f(x) = e^x e^{2x} \). Since the derivative of \( e^x \) is \( e^x \) the derivative of \( e^{2x} \) is \( 2e^{2x} \) and
\[
f(x) = \frac{1}{2} e^x e^{2x} + C.
\]

(9) Find \( f \) such that \( f'(x) = \frac{6x^3 - 2x - 2}{x^2} \).

**Solution:** We have \( \frac{6x^3 - 2x - 2}{x^2} = 6x^2 - \frac{2}{x} - \frac{2}{x^2} \). Since the derivative of \( x^3 \) is \( 3x^2 \), since the derivative of \( \log |x| \) is \( \frac{1}{x} \) and since the derivative of \( \frac{1}{x} \) is \( -\frac{1}{x^2} \) we may use
\[
f(x) = 2x^3 - 2 \log |x| + \frac{2}{x}.
\]

(10) Find \( f \) such that \( f'(x) = 2x^{1/3} - x^{-2/3} \) and \( f(1000) = 5 \).

**Solution:** Since \( (x^{4/3})' = \frac{4}{3}x^{1/3} \) and \( (x^{1/3})' = \frac{1}{3}x^{-2/3} \) the general solutions is
\[
f(x) = 2 \cdot \frac{3}{4} x^{4/3} - 3x^{1/3} + c.
\]

To get the specific solution we solve using \( f(1000) = 5 \):
\[
5 = f(1000) = \frac{3}{2} (1000)^{4/3} - 3(1000)^{1/3} + c
\]
\[
= \frac{3}{2} 10^4 - 30 + c
\]
so
\[
c = 35 - 15,000 = -14,965
\]
and
\[
f(x) = \frac{3}{2} x^{4/3} - 3x^{1/3} - 14,965.
\]

(11) Find \( f \) such that \( f''(x) = \sin x + \cos x \), \( f(0) = 0 \) and \( f'(0) = 1 \).

**Solution:** Since \( (f'(x))' = \sin x + \cos x \), \( f'(x) = -\cos x + \sin x + c \). Now \( f'(0) = -1 + 0 + c = 1 \) so \( c = 2 \) and \( f'(x) = -\cos x + \sin x + 2 \). From this we get \( f(x) = -\sin x - \cos x + 2x + d \) for some \( d \). We also need \( f(0) = -0 - 1 + 0 + d = 0 \) so \( d = 1 \) and
\[
f(x) = -\sin x - \cos x + 2x + 1.
\]