Math 100 – SOLUTIONS TO WORKSHEET 21
OPTIMIZATION

(1) (Final 2012) The right-angled triangle ΔABP has the vertex A = (−1, 0), a vertex P on the semicircle \( y = \sqrt{1 - x^2} \), and another vertex B on the x-axis with the right angle at B. What is the largest possible area of this triangle?

Solution: (0) Picture

(1) Put the coordinate system where the centre of the circle is at (0, 0) and the diameter is on the x-axis. Let B be at \((x, 0)\), P at \((x, y)\).

(2) Since P is on the circle we have \( y = \sqrt{1 - x^2} \). The area of the triangle is then \( A = \frac{1}{2} \text{(base)} \times \text{(height)} = \frac{1}{2}(1 + x)\sqrt{1 - x^2} \) since the base of the triangle has length \(1 + x\).

(4) The function \( A(x) \) is continuous on \([-1, 1]\) so we can find its minimum by differentiation. By the product rule and chain rule,

\[
A'(x) = \frac{1}{2} \sqrt{1 - x^2} + \frac{1}{2}(1 + x) - \frac{2x}{2\sqrt{1 - x^2}}
\]

\[
= \frac{(\sqrt{1 - x^2})^2}{2\sqrt{1 - x^2}} - \frac{x(1 + x)}{2\sqrt{1 - x^2}} = \frac{1 - x^2 - x - x^2}{2\sqrt{1 - x^2}}
\]

\[
= \frac{1 - x - 2x^2}{2\sqrt{1 - x^2}}.
\]

This is defined on \((-1, 1)\) and the critical points satisfy \(2x^2 + x - 1 = 0\) so they are \(x = -\frac{1\pm\sqrt{1+8}}{4} = -\frac{1\pm3}{2} = -1, \frac{1}{2}\). The only critical point in the interior is then \(x = \frac{1}{2}\). The area vanishes at the endpoints (the triangle becomes degenerate) and

\[
A\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \sqrt{1 - \frac{1}{4}} = \frac{3\sqrt{3}}{8}.
\]

It follows that the largest possible area is \(\frac{3\sqrt{3}}{8}\).

(2) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs $40/km to build a bridge across the river, $20/km to build a road along it. What is the cheapest way to construct a path between the cities?
Solution: (0) Picture

(1) Build a road of length \( x \) from \( A \) along the bank, then build a bridge of length \( d \) toward \( B \).

(2) By Pythagoras, \( d = \sqrt{6^2 + (8-x)^2} \).

(3) The total cost is
\[
C(x) = 20x + 40\sqrt{6^2 + (8-x)^2} = 20x + 40\sqrt{6^2 + (x-8)^2}.
\]

(4) The function \( C(x) \) is defined everywhere \((6^2 + (8-x)^2 \geq 6^2 > 0)\) and continuous there. We have
\[
C'(x) = 20 + 40 \frac{8-x}{2\sqrt{6^2 + (x-8)^2}}.
\]

This exists everywhere (the denominator is everywhere positive by the same calculation). It’s enough to consider \( 0 \leq x \leq 8 \) (no point in starting the bridge west of \( A \) or east of \( B \)). Looking for critical points we solve \( C'(x) = 0 \) that is:
\[
20 + 40 \frac{x-8}{\sqrt{36 + (x-8)^2}} = 0
\]
\[
20 = 40 \frac{8-x}{\sqrt{36 + (x-8)^2}}
\]
\[
\sqrt{36 + (x-8)^2} = 2(8-x)
\]
\[
36 + (8-x)^2 = 4(8-x)^2
\]
\[
36 = 3(8-x)^2
\]
\[
(8-x) = \sqrt{\frac{36}{3}} = \sqrt{12} = 2\sqrt{3}
\]
(only the positive root since \( 0 \leq x \leq 8 \) forces \( 8-x \geq 0 \)) so
\[
x = 8 - 2\sqrt{3}.
\]
We then have \( C(0) = 40\sqrt{6^2 + 8^2} = 40\sqrt{100} = 400 \), \( C(8) = 20 \cdot 8 + 40\sqrt{6^2} = 160 + 240 = 400 \) and
\[
C(8-2\sqrt{3}) = 20 \left( 8 - 2\sqrt{3} \right) + 40\sqrt{6^2 + (2\sqrt{3})^2} = 160 - 40\sqrt{3} + 40\sqrt{36 + 12}
\]
\[
= 160 - 40\sqrt{3} + 40\sqrt{48} = 160 - 40\sqrt{3} + 40\sqrt{16 \cdot 3}
\]
\[
= 160 - 40\sqrt{3} + 40 \cdot 4\sqrt{3} = 160 + 120\sqrt{3}.
\]
Now \( \sqrt{3} < \sqrt{4} = 2 \) so \( C(8-2\sqrt{3}) = 160 + 120\sqrt{3} < 160 + 120 \cdot 2 = 400 = C(0) = C(8) \) and we conclude that \( C(8-2\sqrt{3}) \) is the minimum.
(5) The cheapest way to construct a bridge is construct a road of length \((8 - 2\sqrt{3})\) km along the bank from \(A\) toward \(B\), and then bridge from the end of the road to \(B\).