1. Applying the MVT

**Theorem.** Let $f$ be defined and continuous on $[a, b]$, differentiable on $(a, b)$. Then there is $c$ between $a, b$ such that \( \frac{f(b) - f(a)}{b - a} = f'(c) \).

Equivalently, for any $x$ there is $c$ between $a, x$ so that $f(x) = f(a) + f'(c)(x - a)$.

(1) Suppose $f'(x) = \frac{e^x}{x + \pi}$ for $0 \leq x \leq 2$. Give an upper bound for $f(2) - f(0)$.

(2) (Final, 2015) Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at most two solutions.

(3) Suppose $f$ satisfies the hypotheses of the MVT and that $f'(x) > 0$ for all $x \in (a, b)$. Show that \( \frac{f(b) - f(a)}{b - a} > 0 \), and hence that $f(b) > f(a)$. 

\[\text{Date: 5/11/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.}\]
2. The shape of a the graph

(4) Let \( f \) be twice differentiable on \([a, b]\).

(a) Suppose first that \( f(a) = f(b) = 0 \) and that \( f \) is positive somewhere between \( a, b \). Show that there is \( c \) between \( a, b \) so that \( f''(c) < 0 \).

(b) Now let \( f(a), f(b) \) take any values, but suppose \( f''(x) > 0 \) on \((a, b)\). Let \( L : y = mx + n \) be the line through \((a, f(a)), (b, f(b))\). Applying part (a) to \( g(x) = f(x) - (mx + n) \) show that the graph of \( f \) lies below the line \( L \).

Definition. We say \( f \) is concave up (or “convex”) on an interval \([a, b]\) if its graph lies under the secant lines in this interval. This is true, for example, if \( f'' > 0 \) on \((a, b)\). We say \( f \) is concave down (or “concave”) on the interval if its graph lies below the scant lines, in particular when \( f'' < 0 \) on \((a, b)\). We say that \( f \) has an inflection point at \( x_0 \) if its second derivative changes sign there.