Math 100 – WORKSHEET 17
THE MEAN VALUE THEOREM

1. MORE MINIMA AND MAXIMA

(1) Show that the function \( f(x) = 3x^3 + 2x - 1 + \sin x \) has no local maxima or minima. You may use that \( f'(x) = 9x^2 + 2 + \cos x \).

(2) Let \( g(x) = xe^{-x^2/8} \) so that \( g'(x) = \left(1 - \frac{x^2}{4}\right)e^{-x^2/8} \), find the global minimum and maximum of \( g \) on
(a) \([-1, 4]\)  
(b) \([0, \infty)\)

(3) Find the critical numbers and singularities of \( h(x) = \begin{cases} x^3 - 6x^2 + 3x & x \leq 3 \\ \sin(2\pi x) - 18 & x \geq 3 \end{cases} \)

(4) (Final, 2014) Find \( a \) such that \( f(x) = \sin(ax) - x^2 + 2x + 3 \) has a critical point at \( x = 0 \).

2. AVERAGE SLOPE VS INSTANTENOUS SLOPE

(5) Let \( f(x) = e^x \) on the interval \([0, 1]\). Find all values of \( c \) so that \( f'(c) = \frac{f(1) - f(0)}{1 - 0} \).

(6) Let \( f(x) = |x| \) on the interval \([-1, 2]\). Find all values of \( c \) so that \( f'(c) = \frac{f(2) - f(-1)}{2 - (-1)} \).
3. The Mean Value Theorem

**Theorem.** Let $f$ be defined and differentiable on $[a, b]$. Then there is $c$ between $a, b$ such that $f(b) - f(a) = f'(c)(b - a)$.

Equivalently, for any $x$ there is $c$ between $a, x$ so that $f(x) = f(a) + f'(c)(x - a)$.

(7) Show that $f(x) = 3x^3 + 2x - 1 + \sin x$ has exactly one real zero. (Hint: let $a, b$ be zeroes of $f$. The MVT will find $c$ such that $f'(c) =$?)

(8) (Final, 2015)

(a) Suppose $f, f', f''$ are all continuous. Suppose $f$ has at least three zeroes. How many zeroes must $f', f''$ have?

(b) [Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions]

(c) Show that the equation has at most two solutions.

(9) (Final, 2012) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x \in [1, 4]$. What can you say about $f(4)$?

(10) Show that $|\sin a - \sin b| \leq |a - b|$ for all $a, b$.

(11) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1 + x) \leq x$. 

2