Math 100 – SOLUTIONS TO WORKSHEET 17

THE MEAN VALUE THEOREM

1. More minima and maxima

1. Show that the function \( f(x) = 3x^3 + 2x - 1 + \sin x \) has no local maxima or minima. You may use that \( f' (x) = 9x^2 + 2 + \cos x \).

   **Solution:** \( f \) is everywhere differentiable, so it can only have a local extremum at a critical point. But for any \( x \) we have \( f'(x) = 9x^2 + 2 + \cos x \geq 0 + 2 - 1 = 1 > 0 \) so \( f \) has no critical points.

2. Let \( g(x) = xe^{-x^2/8} \) so that \( g'(x) = \left( 1 - \frac{x^2}{4} \right) e^{-x^2/8} \), find the global minimum and maximum of \( g \) on
   
   (a) \([-1, 4]\]
   
   (b) \([0, \infty)\]

   **Solution:** \( g \) is everywhere differentiable, and it has critical points at \( x = \pm 2 \). We now calculate: \( g(-1) = -e^{-1/8} \), \( g(0) = 0 \), \( g(2) = 2e^{-1/2} \), \( g(4) = 4e^{-2} \). First of all

   \[
   g(2) = \frac{2}{\sqrt{e}} > \frac{2}{\sqrt{4}} = \frac{4}{e^2} = g(4)
   \]

   so the maximum on \([-1, 4]\) is \( g(2) = \frac{2}{\sqrt{e}} \), while the minimum there is clearly \( g(-1) = -e^{-1/8} \) being the only negative value among the four. Since the function is positive on \((0, \infty)\) its minimum on \([0, \infty)\) is \( g(0) = 0 \). Now \( g \) is decreasing for \( x > 2 \) (the derivative is negative) so the maximum must occur before then. But then it must be at the critical point 2, so the maximum is \( f(2) = \frac{2}{\sqrt{e}} \).

3. Find the critical numbers and singularities of \( h(x) = \begin{cases} x^3 - 6x^2 + 3x & x \leq 3 \\ \sin(2\pi x) - 18 & x \geq 3 \end{cases} \)

   **Solution:** For \( x \leq 3 \), \( h'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2 \) with a singular point at \( x = 1 \). For \( x \geq 3 \) \( f'(x) = 2\pi \cos(2\pi x) \) with critical points at \( x = 3 + \frac{1}{2} + \frac{1}{2}Z_{\geq 0} \). Also, \( x = 3 \) might be a singular point.

4. (Final, 2014) Find a such that \( f(x) = \sin(ax) - x^2 + 2x + 3 \) has a critical point at \( x = 0 \).

   **Solution:** \( f'(x) = a\cos(ax) - 2x + 2 \) so \( f'(0) = a\cos(0) - 2 \cdot 0 + 2 = a + 2 \) so \( f'(0) = 0 \) if \( a = -2 \).

2. Average slope vs Instantenous slope

5. Let \( f(x) = e^x \) on the interval \([0, 1]\). Find all values of \( c \) so that \( f'(c) = \frac{f(1)-f(0)}{1-0} \).

   **Solution:** \( \frac{f(1)-f(0)}{1-0} = \frac{e^1}{1} = e - 1 \) and \( f'(x) = e^x \) so if \( e^c = e - 1 \) we have \( c = \log(e - 1) \) and indeed \( 1 < e - 1 < e \) means \( 0 < \log(e - 1) < 1 \).

6. Let \( f(x) = |x| \) on the interval \([-1, 2]\). Find all values of \( c \) so that \( f'(c) = \frac{f(2)-f(-1)}{2-(-1)} \)

   **Solution:** There is no such value: \( \frac{f(2)-f(-1)}{2-(-1)} = \frac{2-1}{3} = \frac{1}{3} \) but \( f'(x) \) only takes the values \( \pm 1 \).

3. The Mean Value Theorem

7. Show that \( f(x) = 3x^3 + 2x - 1 + \sin x \) has exactly one real zero. (Hint: let \( a, b \) be zeroes of \( f \). The MVT will find \( c \) such that \( f'(c) = ? \))

   **Solution:** Suppose \( f(a) = f(b) = 0 \). The function \( f \) is everywhere differentiable (defined by formula everywhere), so by the MVT there is \( c \) between \( a, b \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} = 0 \). But we know that \( f'(x) \) is everywhere non-vanishing (see problem (1) above).

8. (Final, 2015)
(a) Suppose $f, f', f''$ are all continuous. Suppose $f$ has at least three zeroes. How many zeroes must $f', f''$ have?

**Solution:** Suppose $f(a) = f(b) = 0$. Since $f$ is everywhere differentiable, by the MVT there is $x$ between $a, b$ such that $f'(x) = \frac{f(b) - f(a)}{b - a} = 0$. Now if $a < b < c$ are zeroes of $f$ we find a zero of $f'$ between $(a, b)$ and between $(b, c)$ (so $f'$ has at least two zeroes) and then $f''$ has a zero between the two zeroes of $f'$, so $f''$ has at least one zero.

(b) [Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions]

(c) Show that the equation has at most two solutions.

**Solution:** Suppose $f(x) = 2x^2 - 3 + \sin x + \cos x$ had three zeroes. Then by part (a), $f''(x)$ would have a zero. But 

$$f''(x) = 4 - \sin x - \cos x \geq 4 - 1 - 1 = 2 > 0$$

is nowhere vanishing.

(9) (Final, 2012) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x \in [1, 4]$. What can you say about $f(4)$?

**Solution:** Since $f$ is everywhere differentiable, by the MVT there is $c \in (1, 4)$ such that

$$\frac{f(4) - f(1)}{4 - 1} = f'(c).$$

It follows that

$$-3 \leq \frac{f(4) - f(1)}{3} \leq 2$$

and hence

$$-6 \leq f(1) + (-3) \cdot 3 \leq f(4) \leq f(1) + 2 \cdot 3 = 9.$$

(10) Show that $|\sin a - \sin b| \leq |a - b|$ for all $a, b$.

**Solution:** The claim is automatic if $a = b$ so assume $a \neq b$. Since $f(x) = \sin x$ is everywhere differentiable, for any $a \neq b$ we may apply the MVT to find $c$ between them such that

$$\frac{\sin a - \sin b}{a - b} = f'(c) = \cos c.$$ 

It follows that

$$\frac{|\sin a - \sin b|}{|a - b|} = |\cos c| \leq 1$$

and the claim follows.

(11) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1 + x) \leq x$.

**Solution:** The function $e^x$ is everywhere differentiable and its derivative is $e^x$. For $x > 0$ we therefore have $0 < c < x$ such that

$$\frac{e^x - e^0}{x - 0} = e^c > 1.$$ 

(the latter since $c > 0$). It follows that $e^x > x + e^0 = x + 1$.

Similarly, the function $\log(y)$ is differentiable on $[1, \infty)$ with derivative $\frac{1}{y}$. It follows that for $x > 0$ we have $d$ in the interval $1 < d < 1 + x$ such that

$$\frac{\log(1 + x) - \log 1}{(1 + x) - 1} = \frac{1}{d} < 1$$

(the latter since $d > 1$). Since $\log 1 = 0$ and $(1 + x) - 1 = x$ it follows that

$$\log(1 + x) \leq x.$$