Math 100 – SOLUTIONS TO WORKSHEET 17
THE MEAN VALUE THEOREM

1. More minima and maxima

(1) Show that the function $f(x) = 3x^3 + 2x - 1 + \sin x$ has no local maxima or minima. You may use that $f'(x) = 9x^2 + 2 + \cos x$.

Solution: f is everywhere differentiable, so it can only have a local extremum at a critical point. But for any x we have $f'(x) = 9x^2 + 2 + \cos x \geq 0 + 2 - 1 = 1 > 0$ so f has no critical points.

(2) Let $g(x) = xe^{-x^2/8}$ so that $g'(x) = \left(1 - \frac{x^2}{4}\right) e^{-x^2/8}$, find the global minimum and maximum of g on

(a) $[-1, 4]$ (b) $[0, \infty)$

Solution: g is everywhere differentiable, and it has critical points at $x = \pm 2$. We now calculate: $g(-1) = -e^{-1/8}$, $g(0) = 0$, $g(2) = 2e^{-1/2}$, $g(4) = 4e^{-2}$. First of all

$$g(2) = \frac{2}{\sqrt{e}} > \frac{2}{\sqrt{4}} = \frac{4}{2^2} > \frac{4}{e^2} = g(4)$$

so the maximum on $[-1, 4]$ is $g(2) = \frac{2}{\sqrt{e}}$, while the minimum there is clearly $g(-1) = -e^{-1/8}$ being the only negative value among the four. Since the function is positive on $(0, \infty)$ its minimum on $[0, \infty)$ is $g(0) = 0$. Now g is decreasing for $x > 2$ (the derivative is negative) so the maximum must occur before then. But then it must be at the critical point 2, so the maximum is $f(2) = \frac{2}{\sqrt{e}}$.

(3) Find the critical numbers and singularities of $h(x) = \begin{cases} x^3 - 6x^2 + 3x & x \leq 3 \\ \sin(2\pi x) - 18 & x \geq 3 \end{cases}$

Solution: For $x \leq 3$, $h'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$ with a singular point at $x = 1$. For $x \geq 3$ $f'(x) = 2\pi \cos(2\pi x)$ with critical points at $x = 3\frac{1}{4} + \frac{1}{2}\mathbb{Z}_{\geq 0}$. Also, $x = 3$ might be a singular point.

(4) (Final, 2014) Find a such that $f(x) = \sin(ax) - x^2 + 2x + 3$ has a critical point at $x = 0$.

Solution: $f'(x) = a\cos(ax) - 2x + 2$ so $f'(0) = a\cos(0) - 2 \cdot 0 + 2 = a + 2$ so $f'(0) = 0$ iff $a = -2$.

2. Average slope vs Instantenous slope

(5) Let $f(x) = e^x$ on the interval $[0, 1]$. Find all values of c so that $f'(c) = \frac{f(1)-f(0)}{1-0}$.

Solution: $\frac{f(1)-f(0)}{1-0} = \frac{e-1}{1} = e - 1$ and $f'(x) = e^x$ so if $e^c = e - 1$ we have $c = \log(e-1)$ and indeed $1 < e - 1 < e$ means $0 < \log(e-1) < 1$.

(6) Let $f(x) = |x|$ on the interval $[-1, 2]$. Find all values of c so that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$

Solution: There is no such value: $\frac{f(2)-f(-1)}{2-(-1)} = \frac{2-1}{3} = \frac{1}{3}$ but $f'(x)$ only takes the values ± 1.

3. The Mean Value Theorem

(7) Show that $f(x) = 3x^3 + 2x - 1 + \sin x$ has exactly one real zero. (Hint: let a, b be zeroes of f. The MVT will find c such that $f'(c) = ?$)

Solution: Suppose $f(a) = f(b) = 0$. The function f is everywhere differentiable (defined by formula everywhere), so by the MVT there is c between a, b such that $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. But we know that $f'(x)$ is everywhere non-vanishing (see problem (1) above).

(8) (Final, 2015)

Date: 1/11/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(a) Suppose f, f', f'' are all continuous. Suppose f has at least three zeroes. How many zeroes must f', f'' have?
Solution: Suppose $f(a) = f(b) = 0$. Since f is everywhere differentiable, by the MVT there is x between a, b such that $f'(x) = \frac{f(b)-f(a)}{b-a} = 0$. Now if $a < b < c$ are zeroes of f we find a zero of f' between (a, b) and between (b, c) (so f' has at least two zeroes) and then f'' has a zero between the two zeroes of f', so f'' has at least one zero.

(b) [Show that $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions]

(c) Show that the equation has at most two solutions.
Solution: Suppose $f(x) = 2x^2 - 3 + \sin x + \cos x$ had three zeroes. Then by part (a), $f''(x)$ would have a zero. But $f''(x) = 4 - \sin x - \cos x \geq 4 - 1 - 1 = 2 > 0$ is nowhere vanishing.

(9) (Final, 2012) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x \in [1, 4]$. What can you say about $f(4)$?
Solution: Since f is everywhere differentiable, by the MVT there is $c \in (1, 4)$ such that

$$\frac{f(4) - f(1)}{4 - 1} = f'(c)\,.$$

It follows that

$$-3 \leq \frac{f(4) - f(1)}{3} \leq 2$$

and hence

$$-6 \leq f(1) + (-3) \cdot 3 \leq f(4) \leq f(1) + 2 \cdot 3 = 9\,.$$

(10) Show that $|\sin a - \sin b| \leq |a - b|$ for all a, b.
Solution: The claim is automatic if $a = b$ so assume $a \neq b$. Since $f(x) = \sin x$ is everywhere differentiable, for any $a \neq b$ we may apply the MVT to find c between them such that $\frac{\sin a - \sin b}{a-b} = f'(c) = \cos c$. It follows that

$$\frac{|\sin a - \sin b|}{|a - b|} = |\cos c| \leq 1$$

and the claim follows.

(11) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1 + x) \leq x$.
Solution: The function e^x is everywhere differentiable and its derivative is e^c. For $x > 0$ we therefore have $0 < c < x$ such that

$$\frac{e^x - e^0}{x - 0} = e^c > 1\,.$$

(the latter since $c > 0$). It follows that $e^x > x + e^0 = x + 1$.
Similarly, the function $\log(y)$ is differentiable on $[1, \infty)$ with derivative $\frac{1}{y}$. It follows that for $x > 0$ we have d in the interval $1 < d < 1 + x$ such that

$$\frac{\log(1 + x) - \log 1}{(1 + x) - 1} = \frac{1}{d} < 1$$

(the latter since $d > 1$). Since $\log 1 = 0$ and $(1 + x) - 1 = x$ it follows that

$$\log(1 + x) \leq x\,.$$