1. Exponentials

(1) Suppose that a pair of invasive opossums arrives in BC in 1935. Unchecked, opossums can triple their population annually.

(a) At what time will there be 1000 opossums in BC? 10,000 opossums?

**Solution:** After \( t \) years there are \( 2 \cdot 3^t \) opossums, so there will be 1000 opossums after \( \frac{\log 500}{\log 3} \approx 5.7 \) years and 10,000 opossums after \( \frac{\log 5000}{\log 3} \approx 7.8 \) years.

(b) Write a differential equation expressing the growth of the opossum population with time.

**Solution:** Since \( \frac{d}{dt}3^t = \log 3 \cdot 3^t \) the differential equation is \( \frac{dN}{dt} = \log 3 \cdot N \).

(2) A radioactive sample decays according to the law \( \frac{dm}{dt} = km \).

(a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?

(b) A 100-gram sample is left unattended for three days. How much of it remains?

**Solution:** If two halvings happened in 10 hours, the half-life is \( \frac{72}{5} \) hours. Accordingly after three days we will have \( 100 \cdot \frac{72}{5} \) half-lives, so the remaining mass will be \( 100 \cdot 2^{-\frac{72}{5}} = 100 \cdot \exp \left( -\frac{72}{5} \log 2 \right) \approx 0.005 \) g.

(3) (Final, 2015) A colony of bacteria doubles every 4 hours. If the colony has 2000 cells after 6 hours, how many were present initially? Simplify your answer.

**Solution:** After 6 hours we’ve had 1.5 doublings, so the original size is \( \frac{2000}{2^{1.5}} = 1000 \cdot \frac{\sqrt{2}}{2} \approx 707 \).

2. Newton’s Law of Cooling

**Fact.** When a body of temperature \( T_0 \) is placed in an environment of temperature \( T_{\text{env}} \), the rate of change of the temperature \( T(t) \) is negatively proportional to the temperature difference \( T - T_{\text{env}} \). In other words, there is a (negative) constant \( k \) such that

\[
T' = k(T - T_{\text{env}}).
\]

**key idea:** change variables to the temperature difference. Let \( y = T - T_{\text{env}} \). Then

\[
\frac{dy}{dt} = \frac{dT}{dt} - 0 = ky
\]

so there is \( C \) for which

\[
y(t) = Ce^{kt}.
\]

Solving for \( T \) we get:

\[
T(t) = T_{\text{env}} + Ce^{kt}.
\]

Setting \( t = 0 \) we find \( T_{\text{env}} + C = T_0 \) so \( C = T_0 - T_{\text{env}} \) and

\[
T(t) = T_{\text{env}} + (T_0 - T_{\text{env}})e^{kt}.
\]

**Corollary.** \( \lim_{t \to \infty} y(t) = 0 \) so \( \lim_{t \to \infty} T(y) = T_{\text{env}} \).

(1) (Final, 2010) When an apple is taken from a refrigerator, its temperature is 3\(^\circ\)C. After 30 minutes in a 19\(^\circ\)C room its temperature is 11\(^\circ\)C.

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Date: 15/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(a) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.

**Solution:** Let \( T(t) \) be the temperature of the apple \( t \) minutes after it was taken from the refrigerator, and let \( y(t) = T(t) - 19 \) be the temperature difference. Newton’s law of cooling provides that \( y(t) \) decays exponentially at constant rate. We are given that \( y(0) = -16^\circ C \) and that \( y(30) = -8^\circ C \) so the temperature difference was halved after 30 minutes. By 90 minutes there would be two further halvings, so \( y(90) = -1^\circ C \) and \( T(90) = 17^\circ C \).

(b) Determine the time when the temperature of the apple is \( 16^\circ C \).

**Solution:** We are asked when the temperature difference will be \( -3^\circ C \). Since the temperature difference satisfies the law \( y(t) = -16^\circ C \cdot 2^{-t/30} \) we need to find \( t \) so that

\[
-3 = -16 \cdot 2^{-t/30}
\]

that is

\[
2^{t/30} = \frac{16}{3}.
\]

Taking logarithms of both sides we have

\[
\frac{t}{30} \log 2 = \log 16 - \log 3
\]

so that the apple reaches \( 16^\circ C \) at time

\[
t = 30 \cdot \frac{\log 16 - \log 2}{\log 2}\text{ minutes.}
\]

(c) Write the differential equation satisfied by the temperature \( T(t) \) of the apple.

**Solution:** We are asked when the temperature difference will be \( -3^\circ C \). Since the temperature difference satisfies the law \( y(t) = -16^\circ C \cdot 2^{-t/30} \) we need to find \( t \) so that

\[
-3 = -16 \cdot 2^{-t/30}
\]

that is

\[
2^{t/30} = \frac{16}{3}.
\]

Taking logarithms of both sides we have

\[
\frac{t}{30} \log 2 = \log 16 - \log 3
\]

so that the apple reaches \( 16^\circ C \) at time

\[
t = 30 \cdot \frac{\log 16 - \log 2}{\log 2}\text{ minutes.}
\]

The temperature of the apple at time \( T \) is \( T(t) = 19^\circ C - 16^\circ C \cdot 2^{-t/30} = T = 19^\circ C - 16^\circ C \cdot e^{-\frac{\log 2}{30} t} \). We therefore have

\[
\frac{dT}{dt} = -\frac{\log 2}{30} (T - 19) .
\]

(2) (Final, 2013) A bottle of soda pop at room temperature \( (70^\circ F) \) is placed in the refrigerator where the temperature is \( 40^\circ F \). After half an hour the bottle has cooled to \( 60^\circ F \). When will it reach \( 50^\circ F \)?

**Solution:** Let \( T(t) \) be the temperature of the soda \( t \) minutes after it was put in the fridge, and let \( y(t) = T(t) - 40^\circ F \) be the temperature difference. Then we are given that \( y(0) = 30^\circ F \) and that \( y(30) = 20^\circ F \). Newton’s law of cooling provides that \( y(t) \) decays exponentially at constant rate, so we conclude that \( y(t) = 30^\circ F \cdot \left(\frac{2}{3}\right)^{t/30} \). We are asked for the time when \( T(t) = 50^\circ F \), that is when \( y(t) = 10^\circ F \). That time therefore satisfies:

\[
10 = 30 \cdot \left(\frac{2}{3}\right)^{t/30},
\]
that is

\[ 3 = \left( \frac{3}{2} \right)^{t/30}. \]

Taking logarithms we find

\[ \log 3 = \frac{t}{30} \log \frac{3}{2} \]

so

\[ t = 30 \frac{\log 3}{\log 3 - \log 2} \text{ minutes.} \]