1. Implicit Differentiation

(1) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point (2, 6).

**Solution:** Differentiating with respect to $x$ we find $2y\frac{dy}{dx} = 12x^2 + 2$, so that $\frac{dy}{dx} = \frac{6x^2 + 1}{y}$. In particular at the point $(2, 6)$ the slope is $\frac{25}{6}$ and the line is

$$y = \frac{25}{6} (x - 2) + 6.$$ 

(2) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{dy}{dx}$ at the point (1, 1).

**Solution:** Differentiating with respect to $x$ we find $y^2 + 2xy\frac{dy}{dx} + 2xy + 2x^2\frac{dy}{dx} = 0$ along the curve. Setting $x = y = 1$ we get that, at the indicated point,

$$3 + 4\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{3}{4}.$$ 

In particular at the point $(2, 6)$ the slope is $\frac{25}{6}$ and the line is

$$y = \frac{25}{6} (x - 2) + 6.$$ 

(3) (Final 2012) Find the slope of the line tangent to the curve $y + x \cos y = \cos x$ at the point (0, 1).

**Solution:** Differentiating with respect to $x$ we find $y' + \cos y - x \sin y \cdot y' = -\sin x$, so that $y' = -\frac{\sin x + \cos y}{1 - x \sin y}$. Setting $x = 0$, $y = 1$ we get that at that point $y' = \frac{\cos 1}{\sin 1} = -\cos 1$.

(4) Find $y''$ (in terms of $x, y$) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).

**Solution:** Differentiating with respect to $x$ we find $5x^4 + 5y^4 y' = 0$, so that $y' = -\frac{x^4}{y^4}$. Differentiating again we find

$$y'' = \frac{4x^3}{y^4} + \frac{4x^4 y'}{y^5} = \frac{4x^3}{y^4} - \frac{4x^8}{y^9}.$$ 

(5) Find $y'$ if $(x + y) \sin(xy) = x^2$.

**Solution:** Differentiating with respect to $x$ we find $(1 + y') \sin(xy) + (x + y) \cos(xy) y + xy = 2x$, so that

$$y' \left[ \sin(xy) + x(x + y) \cos(xy) \right] = 2x - \left[ \sin(xy) + y(x + y) \cos(xy) \right]$$

so that

$$y' = \frac{2x - \sin(xy) - y(x + y) \cos(xy)}{\sin(xy) + x(x + y) \cos(xy)}.$$ 

2. Inverse Trig Functions

(1) Evaluation

(a) (Final 2014) Evaluate $\arcsin \left( -\frac{1}{2} \right)$; Find $\arcsin \left( \frac{3\pi}{11} \right)$.

**Solution:** $\sin \left( \frac{\pi}{6} \right) = \frac{1}{2}$ so $\arcsin \left( -\frac{1}{2} \right) = -\frac{\pi}{6}$. Also $\sin \left( \frac{3\pi}{11} \right) = \sin \left( \frac{3\pi}{11} - 2\pi \right) = \sin \left( \frac{9\pi}{11} \right) = \sin \left( \pi - \frac{2\pi}{11} \right) = \sin \left( \frac{2\pi}{11} \right)$ and $\frac{2\pi}{11} \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$ so $\arcsin \left( \frac{3\pi}{11} \right) = \frac{2\pi}{11}$. 

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*Date: 8/10/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.*
(b) (Final 2015) Simplify $\sin(\arctan 4)$

**Solution:** Consider the right-angled triangle with sides 4, 1 and hypotenuse $\sqrt{1+4^2} = \sqrt{17}$. Let $\theta$ be the angle opposite the side of length 4. Then $\tan \theta = 4$ and $\sin \theta = \frac{4}{\sqrt{17}}$ so $\sin (\arctan 4) = \sin \theta = \frac{4}{\sqrt{17}}$.

(c) Find $\tan (\arccos (0.4))$

**Solution:** Consider the right-angled triangle with sides 0.4, $\sqrt{1-0.4^2}$ and hypotenuse 1. Let $\theta$ be the angle between the side of length 0.4 and the hypotenuse. Then $\cos \theta = 0.4$ and $\tan \theta = \frac{\sqrt{1-0.4^2}}{0.4} = \frac{\sqrt{0.84}}{0.4} = \frac{\sqrt{0.84}}{0.16} = 5.25$.

(2) Differentiation

(a) Find $\frac{d}{dx} (\arcsin (2x))$

**Solution:** Since $\frac{d}{dx} \arcsin (x) = \frac{1}{\sqrt{1-x^2}}$, the chain rule gives $\frac{d}{dx} (\arcsin (2x)) = \frac{2}{\sqrt{1-4x^2}}$. Alternatively, let $\theta = \arcsin 2x$, so that $\sin \theta = 2x$. Differentiating both sides we get $\cos \theta \cdot \frac{d\theta}{dx} = 2$ so that $\frac{d\theta}{dx} = \frac{2}{\cos \theta} = \frac{2}{\sqrt{1-4x^2}}$.

(b) Find the line tangent to $y = \sqrt{1 + (\arctan(x))^2}$ at the point where $x = 1$.

**Solution:** Since $\frac{d}{dx} \arctan (x) = \frac{1}{1+x^2}$, the chain rule gives $\frac{d}{dx} \sqrt{1 + (\arctan(x))^2} = \frac{1}{2 \sqrt{1 + (\arctan(x))^2}} \cdot 2 \arctan(x) \cdot \frac{1}{1+x^2}$

$$= \frac{\arctan x}{(1+x^2) \sqrt{1 + (\arctan(x))^2}}.$$ 

Now $\arctan 1 = \frac{\pi}{4}$ so the line is

$$y = \frac{\pi}{8 \sqrt{1 + \frac{x^2}{16}}} (x-1) + \sqrt{1 + \frac{\pi^2}{16}}.$$

(c) Find $y'$ if $y = \arcsin (e^{5x})$. What is the domain of the functions $y, y'$?

**Solution:** From the chain rule we get $\frac{d}{dx} \arcsin (e^{5x}) = \frac{1}{\sqrt{1-e^{10x}}} \cdot 5e^{5x} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}$. The function $y$ itself is defined when $-1 \leq e^{5x} \leq 1$, that is when $5x \leq 0$, that is when $x \leq 0$. The derivative is defined when $-1 < e^{10x} < 1$, that is when $x < 0$. The point is that since $\sin \theta$ has horizontal tangents at $\pm \frac{\pi}{2}$, $\arcsin x$ has vertical tangents at $\pm 1$.

**Solution:** We can write the identity as $\sin y = e^{5x}$ and differentiate both sides to get $y' \cos y = 5e^{5x}$ so that

$$y' = \frac{5e^{5x}}{\cos y} = \frac{5e^{5x}}{\sqrt{1-\sin^2 y}} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}.$$