1. Implicit Differentiation

(1) Find the line tangent to the curve \( y^2 = 4x^3 + 2x \) at the point \((2, 6)\).

Solution: Differentiating with respect to \( x \) we find \( 2y \frac{dy}{dx} = 12x^2 + 2 \), so that \( \frac{dy}{dx} = \frac{6x^2 + 1}{y} \). In particular at the point \((2, 6)\) the slope is \( \frac{25}{6} \) and the line is

\[
y = \frac{25}{6}(x - 2) + 6.
\]

(2) (Final, 2015) Let \( xy^2 + x^2 = 2 \). Find \( \frac{dy}{dx} \) at the point \((1, 1)\).

Solution: Differentiating with respect to \( x \) we find \( y^2 + 2xy \frac{dy}{dx} + 2x + 2x^2 \frac{dy}{dx} = 0 \) along the curve. Setting \( x = y = 1 \) we find that, at the indicated point,

\[
3 + 4 \frac{dy}{dx} = 0
\]

so

\[
\frac{dy}{dx} = -\frac{3}{4}.
\]

In particular at the point \((2, 6)\) the slope is \( \frac{25}{6} \) and the line is

\[
y = \frac{25}{6}(x - 2) + 6.
\]

(3) (Final 2012) Find the slope of the line tangent to the curve \( y + x \cos y = \cos x \) at the point \((0, 1)\).

Solution: Differentiating with respect to \( x \) we find \( y' + \cos y - x \sin y \cdot y' = -\sin x \), so that \( y' = -\frac{\sin x + \cos y}{1 - x \sin y} \). Setting \( x = 0, y = 1 \) we get that at that point \( y' = \frac{\cos 1}{1} = -\cos 1 \).

(4) Find \( y'' \) (in terms of \( x, y \)) along the curve \( x^5 + y^5 = 10 \) (ignore points where \( y = 0 \)).

Solution: Differentiating with respect to \( x \) we find \( 5x^4 + 5y^4 y' = 0 \), so that \( y' = -\frac{x^4}{y^4} \). Differentiating again we find

\[
y'' = \frac{4x^3}{y^4} + \frac{4x^4 y'}{y^5} = \frac{4x^3}{y^4} - \frac{4x^8}{y^9}.
\]

(5) Find \( y' \) if \( (x + y) \sin(xy) = x^2 \).

Solution: Differentiating with respect to \( x \) we find \( (1 + y') \sin(xy) + (x + y) \cos(xy)(y + xy') = 2x \), so that

\[
y' [\sin(xy) + x(x + y) \cos(xy)] = 2x - [\sin(xy) + y(x + y) \cos(xy)]
\]

so that

\[
y' = \frac{2x - \sin(xy) - y(x + y) \cos(xy)}{\sin(xy) + x(x + y) \cos(xy)}.
\]

2. Inverse Trig Functions

(1) Evaluation

(a) (Final 2014) Evaluate \( \arcsin \left( -\frac{1}{2} \right) \); Find \( \arcsin \left( \frac{31\pi}{11} \right) \).

Solution: \( \sin \left( \frac{\pi}{6} \right) = \frac{1}{2} \) so \( \arcsin \left( -\frac{1}{2} \right) = -\frac{\pi}{6} \). Also \( \sin \left( \frac{31\pi}{11} \right) = \sin \left( \frac{31\pi}{11} - 2\pi \right) = \sin \left( \frac{9\pi}{11} \right) = \sin \left( \pi - \frac{2\pi}{11} \right) = \sin \left( \frac{2\pi}{11} \right) \) and \( \frac{2\pi}{11} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \) so \( \arcsin \left( \frac{31\pi}{11} \right) = \frac{2\pi}{11} \).
(b) (Final 2015) Simplify \( \sin(\arctan 4) \)

**Solution:** Consider the right-angled triangle with sides 4, 1 and hypotenuse \( \sqrt{1+4^2} = \sqrt{17} \). Let \( \theta \) be the angle opposite the side of length 4. Then \( \tan \theta = 4 \) and \( \sin \theta = \frac{4}{\sqrt{17}} \) so \( \sin(\arctan 4) = \sin \theta = \frac{4}{\sqrt{17}} \).

(c) Find \( \tan(\arccos(0.4)) \)

**Solution:** Consider the right-angled triangle with sides 0.4, 1 - 0.4 and hypotenuse 1. Let \( \theta \) be the angle between the side of length 0.4 and the hypotenuse. Then \( \cos \theta = \frac{0.4}{1} = 0.4 \) and \( \tan \theta = \frac{1 - 0.4^2}{0.4} = \frac{0.84}{0.4} = \frac{8.4}{4} = 2.1 \).

(2) Differentiation

(a) Find \( \frac{d}{dx}(\arcsin(2x)) \)

**Solution:** Since \( \frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \), the chain rule gives \( \frac{d}{dx}(\arcsin(2x)) = \frac{2}{\sqrt{1-4x^2}} \).

Alternatively, let \( \theta = \arcsin 2x \), so that \( \sin \theta = 2x \). Differentiating both sides we get

\[
\cos \theta \cdot \frac{d\theta}{dx} = 2
\]

so that

\[
\frac{d\theta}{dx} = \frac{2}{\cos \theta} = \frac{2}{\sqrt{1-\sin^2 \theta}} = \frac{2}{\sqrt{1-4x^2}}.
\]

(b) Find the line tangent to \( y = \sqrt{1 + (\arctan(x))^2} \) at the point where \( x = 1 \).

**Solution:** Since \( \frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2} \), the chain rule gives

\[
\frac{d}{dx}\sqrt{1 + (\arctan(x))^2} = \frac{1}{2\sqrt{1 + (\arctan(x))^2}} \cdot 2 \arctan(x) \cdot \frac{1}{1+x^2}
\]

\[
= \frac{\arctan x}{(1+x^2)\sqrt{1 + (\arctan(x))^2}}.
\]

Now \( \arctan 1 = \frac{\pi}{4} \) so the line is

\[
y = \frac{\pi}{8\sqrt{1 + \frac{\pi^2}{16}}} (x - 1) + \sqrt{1 + \frac{\pi^2}{16}}.
\]

(c) Find \( y' \) if \( y = \arcsin(e^{5x}) \). What is the domain of the functions \( y, y' \)?

**Solution:** From the chain rule we get

\[
\frac{d}{dx}(\arcsin(e^{5x})) = \frac{1}{\sqrt{1-e^{10x}}} \cdot 5e^{5x} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}.
\]

The function \( y \) itself is defined when \(-1 \leq e^{5x} \leq 1\), that is when \(5x \leq 0\), that is when \(x \leq 0\).

The derivative is defined when \(-1 < e^{10x} < 1\), that is when \(x < 0\). The point is that since \( \sin \theta \) has horizontal tangents at \( \pm \frac{\pi}{2} \), \( \arcsin x \) has vertical tangents at \( \pm 1 \).

**Solution:** We can write the identity as \( \sin y = e^{5x} \) and differentiate both sides to get \( y' \cos y = 5e^{5x} \) so that

\[
y' = \frac{5e^{5x}}{\cos y} = \frac{5e^{5x}}{\sqrt{1-\sin^2 y}} = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}.
\]