Math 100 – SOLUTIONS TO WORKSHEET 8
INVERSE FUNCTIONS

1. More on the chain rule

(1) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.
Solution: Applying the chain rule we have $f'(g(x)) \cdot g'(x) = 3x^2$. Plugging in $x = 4$ we get $5g'(4) = 3 \cdot 4^2$ and hence $g'(4) = \frac{48}{5}$.

2. Inverse functions

(1) Find the function inverse to $y = x^7 + 3$.
Solution: If $y = x^7 + 3$ then $x^7 = y - 3$ so $x = (y - 3)^{1/7}$, and the inverse function is $y = (x - 3)^{1/7}$.

(2) Does $y = x^2$ have an inverse?
Solution: Not on its full domain (not single-valued), yes on $[0, \infty)$.

(3) Consider the function $y = \sqrt{x - 1}$ on $x \geq 1$.
(a) Find the inverse function, in the form $x = g(y)$.
Solution: If $y = \sqrt{x-1}$ then $y^2 = x - 1$ so $x = y^2 + 1$.
(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}x}{\mathrm{d}y}$ and calculate their product.
Solution: We have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x-1}}$ and $\frac{\mathrm{d}x}{\mathrm{d}y} = 2y$. Their product is $\frac{2y}{2\sqrt{x-1}} = \frac{y}{\sqrt{x-1}} = 1$ since $y = \sqrt{x-1}$ along the curves.

3. The inverse function fule

(1) Given that $\frac{\mathrm{d}}{\mathrm{d}y} y^2 = 2y$, find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}$.
Solution: If $y = \sqrt{x}$ then $x = y^2$ so $\frac{\mathrm{d}x}{\mathrm{d}y} = 2y = 2\sqrt{x}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$ so the derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$.

(2) Find $\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x$.
Solution: Suppose $\theta = \arcsin x$. Then $x = \sin\theta$ so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos\theta$ so $\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{\cos\theta}$ so $\frac{d \arcsin x}{\mathrm{d}x} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}$.

(3) Find $\frac{\mathrm{d}}{\mathrm{d}x} \log x$.
Solution: If $y = \log x$ then $x = e^y$, so $\frac{\mathrm{d}x}{\mathrm{d}y} = e^y$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y} = \frac{1}{x}$.

(4) (Derivatives and logarithms)
(a) Differentiate $\log \sqrt[k]{t}$.
Solution: $\log \sqrt[k]{t} = \log t^{1/k} = \frac{1}{k} \log t$ so the derivative is $\frac{1}{kt}$. We can also use the chain rule:

$$\frac{\mathrm{d} \log t^{1/k}}{\mathrm{d}t} = \frac{1}{t^{1/k}} \cdot \frac{1}{k} t^{\frac{1}{k}-1} = \frac{1}{kt} .$$

(b) (Final, 2012) Let $y = \log(\sin(\log x))$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.
Solution: By the chain rule this is $\frac{1}{\sin(\log x)} \cdot \cos(\log x) \cdot \frac{1}{x} = \frac{\cos(\log x)}{x \sin(\log x)}$.

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