Math 100 – SOLUTIONS TO WORKSHEET 7
TRIGONOMETRIC FUNCTIONS; THE CHAIN RULE

1. Trigonometric Functions

(1) (Special values) What is $\sin\frac{\pi}{3}$? What is $\cos\frac{5\pi}{2}$?
Solution: $\sin\frac{\pi}{3} = \frac{1}{2}$, $\cos\left(\frac{5\pi}{2}\right) = \cos\left(\frac{\pi}{2} + 2\pi\right) = \cos\left(\frac{\pi}{2}\right) = 0$.

(2) Derivatives of trig functions
 (a) Interpret $\lim_{h\to 0}\frac{\sin h}{h}$ as a derivative and find its value.
 Solution: This is $\lim_{h\to 0}\frac{\sin(0+h)-\sin 0}{h} = \frac{d\sin x}{dx}\Big|_{x=0} = \cos 0 = 1$.
 (b) Differentiate $\tan\theta = \frac{\sin\theta}{\cos\theta}$.
 Solution: Applying the quotient rule we get
$$\begin{aligned}\frac{\mathrm{d}\tan\theta}{\mathrm{d}\theta} &= \frac{\cos\theta\cdot\cos\theta - \sin\theta\cdot(-\cos\theta)}{\cos^2\theta}\\ &= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}\,.\end{aligned}$$
 We also have
$$\frac{\mathrm{d}\tan\theta}{\mathrm{d}\theta} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = 1+\tan^2\theta$$
 which is sometimes useful.
 (c) What is the equation of the line tangent the graph $y = T\sin x + \cos x$ at the point where $x = \frac{\pi}{4}$? Here T is a parameter (=constant).
 Solution: We have $y(\frac{\pi}{4}) = \frac{T}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{T+1}{\sqrt{2}}$. Also, $\frac{\mathrm{d}y}{\mathrm{d}x} = T\cos x - \sin x$ so $\frac{\mathrm{d}y}{\mathrm{d}x}|_{x=\frac{\pi}{4}} = \frac{T}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{T-1}{\sqrt{2}}$. So the line is
$$y = \frac{T-1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) + \frac{T+1}{\sqrt{2}}\,.$$

2. The Chain Rule

(1) Write the function as a composition and then differentiate.
 (a) e^{3x}
 Solution: This is $f(g(x))$ where $g(x) = 3x$ and $f(y) = e^x$. The derivative is thus
$$e^{3x}\cdot\frac{\mathrm{d}(3x)}{\mathrm{d}x} = 3e^{3x}\,.$$
 (b) $\sqrt{2x+1}$
 Solution: This is $f(g(x))$ where $g(x) = 2x+1$ and $f(y) = \sqrt{y}$. Thus
$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = f'(g(x))g'(x) = \frac{1}{2\sqrt{g}}\cdot 2 = \frac{1}{\sqrt{2x+1}}\,.$$
 (c) (Final, 2015) $\sin(x^2)$
 Solution: This is $f(g(x))$ where $g(x) = x^2$ and $f(y) = y^2$. The derivative is then
$$\cos(x^2)\cdot 2x = 2x\cos(x^2)\,.$$
 (d) $(7x+\cos x)^n$.
 Solution: This is $f(g(x))$ where $g(x) = 7x+\cos x$ and $f(y) = y^n$. The derivative is thus
$$n\,(7x+\cos x)^{n-1}\cdot(7-\sin x)\,.$$

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(2) Differentiate

(a) $7x + \cos(x^n)$

Solution: We apply linearity and then the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(7x+\cos(x^n)\right) &= \frac{\mathrm{d}(7x)}{\mathrm{d}x} + \frac{\mathrm{d}\cos(x^n)}{\mathrm{d}x} \\ &= 7 + \frac{\mathrm{d}\cos(x^n)}{\mathrm{d}(x^n)} \cdot \frac{\mathrm{d}(x^n)}{\mathrm{d}x} \\ &= 7 - \sin(x^n)\cdot nx^{n-1}\,.\end{aligned}$$

(b) $e^{\sqrt{\cos x}}$

Solution: We repeatedly apply the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} e^{\sqrt{\cos x}} &= e^{\sqrt{\cos x}} \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\cos x} \\ &= e^{\sqrt{\cos x}} \frac{1}{2\sqrt{\cos x}} \frac{\mathrm{d}}{\mathrm{d}x}\cos x \\ &= -e^{\sqrt{\cos x}} \frac{\sin x}{2\sqrt{\cos x}}\,.\end{aligned}$$

(c) (Final 2012) $e^{(\sin x)^2}$

Solution: By the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\sin x)^2}\right) &= e^{(\sin x)^2} \frac{\mathrm{d}}{\mathrm{d}x}\left((\sin x)^2\right) \\ &= e^{(\sin x)^2} 2\sin x \frac{\mathrm{d}}{\mathrm{d}x}\sin x \\ &= e^{(\sin x)^2} 2\sin x\cos x \\ &= e^{(\sin x)^2}\sin(2x).\end{aligned}$$