Math 100 – SOLUTIONS TO WORKSHEET 6
POLYNOMIALS AND EXPONENTIALS

1. Direct problems

(1) Differentiate

(a) $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$

Solution: This is a linear combination of power laws so $f'(x) = 6\pi x^{\pi-1} + 2ex^{e-1} - \frac{7}{2}x^{7/2}$.

(b) (Final, 2016) $f(x) = x^2e^x$ (also try x^ae^x)

Solution: Applying the product rule we get $\frac{df}{dx} = \frac{d(x^2)}{dx} \cdot e^x + x^2 \cdot \frac{d(e^x)}{dx} = (2x + x^2)e^x = x(x+2)e^x$, and in general

$$\frac{d}{dx}(x^ae^x) = ax^{a-1}e^x + x^ae^x = x^{a-1}(x+a)e^x \,.$$

(c) (Final, 2016) $f(x) = \frac{x^2+3}{2x-1}$

Solution: Applying the quotient rule the derivative is $\frac{2x\cdot(2x-1)-(x^2+3)\cdot 2}{(2x-1)^2} = \frac{4x^2-2x-2x^2-6}{(2x-1)^2} = 2\frac{x^2-x-3}{(2x-1)^2}$.

(d) $f(x) = \frac{\sqrt{x}(1-3x)}{x^2+1}$

Solution: We apply the quotient rule to $f(x) = \frac{x^{1/2}-3x^{3/2}}{x^2+1}$ to get:

$$\begin{aligned} f'(x) &= \frac{\left(\frac{1}{2}x^{-1/2} - \frac{9}{2}x^{1/2}\right)\left(x^2+1\right) - \left(x^{1/2} - 3x^{3/2}\right)2x}{\left(x^2+1\right)^2} \\ &= \frac{(1-9x)\left(x^2+1\right) - 2\left(2x - 6x^2\right)x}{2\sqrt{x}\left(x^2+1\right)^2} \\ &= \frac{3x^3 - 3x^2 - 9x + 1}{2\sqrt{x}\left(x^2+1\right)^2} \end{aligned}$$

(e) $f(x) = \frac{x^2+xe^x}{\cos x + \sin x}$

Solution: Apply the quotient and product rules.

$$f'(x) = \frac{(2x + e^x + xe^x)(\cos x + \sin x) - (x^2 + xe^x)(\cos x - \sin x)}{(\cos x + \sin x)^2} \,.$$

2. Exponentials

(1) Simplify $\left(e^5\right)^3$, $\left(2^{1/3}\right)^{12}$, 7^{3-5}.

Solution: $\left(e^5\right)^3 = e^{5\cdot 3} = e^{15}$, $\left(2^{1/3}\right)^{12} = 2^{\frac{1}{3}\cdot 12} = 2^4 = 16$, $7^{3-5} = 7^{-2} = \frac{1}{49}$.

(2) Differentiate:

(a) 10^x

Solution: This is $(\log 10)\cdot 10^x$.

(b) $\frac{5\cdot 10^x + x^2}{3^x+1}$

Solution: By the quotient rule this is

$$\frac{(5\log 10 \cdot 10^x + 2x)\left(3^x+1\right) - \left(5\cdot 10^x + x^2\right)\log 3 \cdot 3^x}{\left(3^x+1\right)^2} \,.$$

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3. Tangent lines

(1) Suppose that $f(1)=1$, $g(1)=2$, $f'(1)=3$, $g'(1)=4$. Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

Solution: $(fg)'(1) = f'(1)g(1)+f(1)g'(1) = 3\cdot 2 + 1\cdot 4 = 10$.

$$\left(\frac{f}{g}\right)'(1) = \frac{f'(1)g(1)-f(1)g'(1)}{(g(1))^2} = \frac{3\cdot 2-1\cdot 4}{2^2} = \frac{1}{2}\,.$$

(2) (Final, 2015) Find the equation of the line tangent to the function $f(x)=\sqrt{x}$ at $(4,2)$.

Solution: $f'(x)=\frac{1}{2\sqrt{x}}$, so the slope of the line is $f'(4)=\frac{1}{4}$, and the equation for the line line itself is $y-2=\frac{1}{4}(x-4)$ or $y=\frac{1}{4}(x-4)+2$ or $y=\frac{1}{4}x+1$.

(3) Find the lines of slope 3 tangent the curve $y=x^3+4x^2-8x+3$.

Solution: $\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+8x-8$, so the line tangent at (x,y) has slope 3 iff $3x^2+8x-8=3$, that is iff $3(x^2-1)+8(x-1)=0$. We can factor this as $(x-1)(3x+11)=0$ so the x-coordinates of the points of tangency are $1, -\frac{11}{3}$ and the lines are:

$$y = 3(x-1)$$
$$y = 3(x+\frac{11}{3}) + \left(\left(\frac{11}{3}\right)^3 + 4\left(\frac{11}{3}\right)^2 - 8\left(\frac{11}{3}\right) + 3\right).$$

(4) Let $f(x)=\frac{g(x)}{x}$, where $g(x)$ is differentiable at $x=1$. The line $y=2x-1$ is tangent to the graph $y=f(x)$ at $x=1$. Find $g(1)$ and $g'(1)$.

Solution: At $x=1$ the line meets the graph of $y=f(x)$ so $2\cdot 1-1=1=f(1)=\frac{g(1)}{1}$ and we concldue that $g(1)=1$. The slope of the line there is 2, so $f'(1)=2$. Since we have

$$f'(x)=\frac{xg'(x)-g(x)x}{x^2}$$

we have $2=f'(1)=g'(1)-g(1)$ so $g'(1)=2+g(1)=3$.

(5) (Final 2015) The line $y=4x+2$ is tangent at $x=1$ to which function: x^3+2x^2+3x, x^2+3x+2, $2\sqrt{x+3}+2$, x^3+x^2-x, x^3+x+2, none of the above?

Solution: The line has slope 4 and meets the curve at $(1,6)$. The last two functions don't evaluate to 6 at 1.We differentiate the first three.

$$\frac{d}{dx}|_{x=1}\left(x^3+2x^2+3x\right) = \left(3x^2+4x+3\right)|_{x=1} = 10$$
$$\frac{d}{dx}|_{x=1}\left(x^2+3x+2\right) = (2x+3)\,|_{x=1} = 5$$
$$\frac{d}{dx}|_{x=1}\left(2\sqrt{x+3}+2\right) = \left(\frac{2}{2\sqrt{x+3}}\right)|_{x=1} = \frac{1}{2}\,.$$

The answer is "none of the above".