Math 100 – SOLUTIONS TO WORKSHEET 4
CONTINUITY: THE IVT; THE DERIVATIVE

1. Continuity

(1) Find c, d, e as appropriate such that each function is continuous on its domain:

Solution: The first function is already continuous on $[0,1)$ and $(1,\infty)$. We have $\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\sqrt{x} = 1$ so we must have $c = 1$. We also need $1 = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(d - x^2\right) = d - 1$ so we need $d = 2$. The second function is already continuous on $(-\infty, 1)$ and $(1,\infty)$. We have $\lim_{x\to 1^-} f(x) = 2\cdot 1^3 - e = 2 - e$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} e\cdot 1^2 + 3 = 3 + e = f(1)$ so the function will be continuous at $x = 1$ iff $2 - e = 3 + e$ that is if $e = \frac{5}{2}$.

(2) Where are the following functions continuous?

Solution: $\left(-\sqrt{7}, \sqrt{7}\right)$, $(-\infty, +\infty)$, $(-\infty, -1)\cup(-1,\infty)$ respectively.

(3) (Final 2011) Suppose f, g are continuous such that $g(3) = 2$ and $\lim_{x\to 3}(xf(x) + g(x)) = 1$. Find $f(3)$.

Solution: Since f, g are continuous and applying the limit laws we have

$$\begin{aligned} 1 = \lim_{x\to 3}(xf(x) + g(x)) &= \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} f(x)\right) + \left(\lim_{x\to 3} g(x)\right) \\ &= 3f(3) + g(3) = 3f(3) + 2\,. \end{aligned}$$

Solving for $f(3)$ we get

$$\boxed{f(3) = -\frac{1}{3}}.$$

2. The Intermediate Value Theorem

Theorem. *Let $f(x)$ be continuous for $a \le x \le b$. Then $f(x)$ takes every value between $f(a), f(b)$.*

(1) Show that:

(a) $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \le x \le 1$.

Solution: f is continuous on $[0,1]$ (given by formula there). We have $f(0) = 1$, $f(1) = -2$. By the intermediate value theorem there is $x_0 \in (0,1)$ such that $f(x_0) = 0 \in (-2,1)$.

(b) $\sin x = x + 1$ has a solution.

Solution: Let $f(x) = x + 1 - \sin x$, so we want x such that $f(x) = 0$. The function f is continuous. Note that $f(100) = 101 - \sin 100 \ge 100$ while $f(-100) = -100 + 1 - \sin 100 \le -98$. By the IVT there is $x_0 \in (-100, 100)$ where $f(x_0) = 0$, that is $x_0 + 1 - \sin x_0 = 0$ so $x_0 + 1 = \sin x_0$.

(2) (Final 2011) Let $y = f(x)$ be continuous with domain $[0,1]$ and range in $[3,5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.

Solution: Consider the diffference $g(x) = f(x) - (2x + 3)$. By arithmetic of limits this is a continuous function. We have $g(0) = f(0) - 3 \ge 3 - 3 = 0$ (since $f(0) \ge 3$). We have $g(1) = f(1) - 5 \le 5 - 5 = 0$. By the IVT $g(x)$ takes every value between $g(0)$ and $g(1)$, so there is x_0 such that $g(x_0) = 0$ and then $f(x_0) - (2x_0 + 3) = 0$ so $f(x_0) = 2x_0 + 3$ so the graphs intersect at the point $(x_0, 2x_0 + 3)$.

(3) (Final 2015) Show that the equation $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions.

Date: 17/9/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Solution: We have $f(0) = 0 - 3 + 0 + 1 = -2$. On the other hand if x has large magnitude then $f(x)$ is positive:

$$f(10) = 200 - 3 + \sin 10 + \cos 10 \geq 200 - 5 = 195\,.$$
$$f(-10) = 200 - 3 - \sin 10 + \cos 10 \geq 200 - 5 = 195\,.$$

Thus $f(-10), f(10)$ are positive, $f(0)$ is negative. Since f is continuous everywhere (given by formula), the IVT shows that its graph crosses the x axis once in $(-10, 0)$ and once in $(0, 10)$.

3. Definition of the derivative

Definition. $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

(1) Find $f'(a)$ if
  (a) $f(x) = x^2$, $a = 3$.
  Solution: $\lim_{h\to 0} \frac{(3+h)^2-(3)^2}{h} = \lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0}(6+h) = 6$.
  (b) $f(x) = \frac{1}{x}$, any a.
  Solution: $\lim_{h\to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{a-(a+h)}{a(a+h)}\right) = \lim_{h\to 0} \frac{-h}{h\cdot a(a+h)} = -\lim_{h\to 0} \frac{1}{a(a+h)} = -\frac{1}{a^2}$.
  (c) $f(x) = x^3 - 2x$, any a. (you may use $(a+h)^3 = a^3 + 3a^2h + 3ah^2 + h^3$).
  Solution: We have

$$\begin{aligned}\frac{(a+h)^3 - 2(a+h) - a^3 + 2a}{h} &= \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 2a - 2h - a^3 + 2a}{h} \\ &= \frac{3a^2h + 3ah^2 + h^3 - 2h}{h} \\ &= 3a^2 - 2 + 3ah + h^2 \xrightarrow[h\to 0]{} 3a^2 - 2\,.\end{aligned}$$

(2) Express the limit as a derivative: $\lim_{h\to 0} \frac{\cos(5+h)-\cos 5}{h}$.
  Solution: This is the derivative of $f(x) = \cos x$ at the point $a = 5$.