Math 100 – SOLUTIONS TO WORKSHEET 3
LIMITS AT INFINITY; CONTINUITY

1. The Squeeze Theorem

(1) $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$.

Solution: Since $-1 \le \sin\theta \le 1$ for all θ while $x^2 \ge 0$ we have for all x that

$$-x^2 \le x^2 \sin\left(\frac{\pi}{x}\right) \le x^2 .$$

Now $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0}(-x^2) = 0$, so by the sandwich theorem $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$ too.

(2) (Final, 2014) Suppose that $8x \le f(x) \le x^2 + 16$ for all $x \ge 0$. Find $\lim_{x\to 4} f(x)$.

Solution: We have $\lim_{x\to 4} 8x = 32$ and $\lim_{x\to 4} x^2 + 16 = 32$ so by the sandwich theorem $\lim_{x\to 4} f(x)$ exists and equals 32.

2. Limits at infinity

(1) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3} =$

Solution: $\lim_{x\to\infty} \frac{x^2+1}{x-3} = \lim_{x\to\infty} \frac{x^2}{x} \cdot \frac{1+\frac{1}{x^2}}{1-\frac{3}{x}} = \lim_{x\to\infty} x \cdot \frac{1+\frac{1}{x^2}}{1-\frac{3}{x}} = \infty.$

(b) (Final, 2015) $\lim_{x\to\infty} \frac{x+1}{x^2+2x-8} =$

Solution: $\lim_{x\to\infty} \frac{x+1}{x^2+2x-8} = \lim_{x\to\infty} \frac{x}{x^2} \cdot \frac{1+\frac{1}{x}}{1+\frac{2}{x}-\frac{8}{x^2}} = \lim_{x\to\infty} \frac{1}{x} \cdot \frac{1+\frac{1}{x}}{1+\frac{2}{x}-\frac{8}{x^2}} = 0.$

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} =$

Solution: We have

$$\begin{aligned}\frac{3x}{\sqrt{4x^2+x}-2x} &= \frac{3x}{\sqrt{x^2\left(4+\frac{x}{x^2}\right)}-2x} = \frac{3x}{\sqrt{x^2}\sqrt{4+\frac{1}{x}}-2x} \\ &= \frac{3x}{|x|\sqrt{4+\frac{1}{x}}-2x} = \frac{3x}{(-x)\sqrt{4+\frac{1}{x}}-2x} \\ &= \frac{3}{-\sqrt{4+\frac{1}{x}}-2} \xrightarrow[x\to-\infty]{} \frac{3}{-\sqrt{4+0}-2} = \boxed{-\frac{3}{4}}.\end{aligned}$$

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(d) $\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} =$

Solution: $\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \lim_{x\to\infty} \frac{x^2\sqrt{1+\frac{\sin x}{x^4}}}{x^2\left(1-\frac{\cos x}{x^2}\right)} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{\sin x}{x^4}}}{1-\frac{\cos x}{x^2}}$. Now for all x we have $-\frac{1}{x^4} \le \frac{\sin x}{x^4} \le \frac{1}{x^4}$ and $-\frac{1}{x^2} \le \frac{\cos x}{x^2} \le \frac{1}{x^2}$. Since $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x^4} = 0$ by the squeeze theorem we have $\lim_{x\to\infty} \frac{\sin x}{x^4} = \lim_{x\to\infty} \frac{\cos x}{x^2} = 0$. Thus

$$\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \frac{\sqrt{1+0}}{1-0} = \boxed{1}.$$

(e) $\lim_{x\to-\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-1}\right) =$

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Solution: We have

$$\begin{aligned}\sqrt{x^2+2x}-\sqrt{x^2-1} &= \sqrt{x^2+2x}-\sqrt{x^2-1}\cdot\frac{\sqrt{x^2+2x}+\sqrt{x^2-1}}{\sqrt{x^2+2x}+\sqrt{x^2-1}} = \frac{\left(x^2+2x\right)-\left(x^2-1\right)}{\sqrt{x^2+2x}+\sqrt{x^2-1}} \\ &= \frac{2x+1}{|x|\sqrt{1+\frac{2}{x}}+|x|\sqrt{1-\frac{1}{x^2}}} = \frac{x\left(2+\frac{1}{x}\right)}{-x\sqrt{1+\frac{2}{x}}+(-x)\sqrt{1-\frac{1}{x^2}}} \\ &= -\frac{2+\frac{1}{x}}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x^2}}} \xrightarrow[x\to-\infty]{} -\frac{2}{1+1} = \boxed{-1}.\end{aligned}$$

3. Continuity

(1) Which of these functions are continuous everywhere? Why?

(a) $f(x)=\begin{cases} x & x<0 \\ \cos x & x\geq 0\end{cases}$

Solution: This is clearly continuous on $(-\infty,0)$ and $(0,\infty)$ but $\lim_{x\to 0^-} f(x)=0$ while $\lim_{x\to 0^+} f(x)=1$.

(b) $f(x)=\begin{cases} x & x<0 \\ \sin x & x\geq 0\end{cases}$

Solution: This is clearly continuous on $(-\infty,0)$ and $(0,\infty)$. Also, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=0=f(0)$ so the function is continuous everywhere.

(2) Let $f(x)=\frac{x^3-x^2}{x-1}$.

(a) Why is $f(x)$ discontinuous at $x=1$?

Solution: The formula is undefined there.

(b) Find b such that $g(x)=\begin{cases} f(x) & x\neq 1 \\ b & x=1\end{cases}$ is continouous everywhere.

Solution: $\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^2(x-1)}{x-1}=\lim_{x\to 1}x^2=1$ so setting $b=1$ gives the desired function.

(c) Find c,d such that $f(x)=\begin{cases} \sqrt{x} & 0\leq x<1 \\ c & x=1 \\ d-x^2 & x>1\end{cases}$ is continuous.

Solution: The function is already continuous on $(-\infty,1)$ and $(1,\infty)$. We have $\lim_{x\to 1^-} f(x)=\lim_{x\to 1}\sqrt{x}=1$ so we must have $c=1$. We also need $1=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\left(d-x^2\right)=d-1$ so we need $d=2$.

(d) (Final 2013) For which value of the constant c is $f(x)=\begin{cases} cx^2+3 & x\geq 1 \\ 2x^3-c & x<1\end{cases}$ continuous on $(-\infty,\infty)$?

Solution: The function is already continuous on $(-\infty,1)$ and $(1,\infty)$. We have $\lim_{x\to 1^-} f(x)=2\cdot 1^3-c=2-c$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} c\cdot 1^2+3=3+c=f(1)$ so the function will be continuous at $x=1$ iff $2-c=3+c$ that is if $c=\frac{5}{2}$.

(3) Where are the following functions continuous?

(a) $\frac{1}{\sqrt{7-x^2}}$

Solution: $\left(-\sqrt{7},\sqrt{7}\right)$.

(b) $\frac{x^2+2x+1}{2+\cos x}$

Solution: $(-\infty,\infty)$

(c) $\frac{2+\cos x}{x^2+2x+1}$

Solution: The denominator vanishes at $x=1$ where the function blows up $(2+\cos 1\neq 0)$ so the domain is $(-\infty,-1)\cup(-1,\infty)$.

(d) $\log(\sin x)$

Solution: The logarithm is defined only for positive arguments, so the domain is $\{x \mid \sin x > 0\} = \bigcup_{k\in\mathbb{Z}} \left(2\pi k - \frac{\pi}{2}, 2\pi k + \frac{\pi}{2}\right)$

(4) (Final 2011) Suppose f, g are continuous such that $g(3) = 2$ and $\lim_{x\to 3}\left(xf(x) + g(x)\right) = 1$. Find $f(3)$.

Solution: Since f, g are continuous and applying the limit laws we have

$$\begin{aligned} 1 = \lim_{x\to 3}\left(xf(x) + g(x)\right) &= \left(\lim_{x\to 3} x\right)\left(\lim_{x\to 3} f(x)\right) + \left(\lim_{x\to 3} g(x)\right) \\ &= 3f(3) + g(3) = 3f(3) + 2\,. \end{aligned}$$

Solving for $f(3)$ we get

$$\boxed{f(3) = -\frac{1}{3}}.$$