9. Logarithms & Logarithmic Differentiation   3/10/2019

Goals: (1) Logarithm Laws

(2) Logarithmic Diff

Last time:

(1) Inverse functions: $f(x) = x^7+3$ is inverse to $g(y) = (y-3)^{1/7}$ (restriction of domain)

(2) Inverse function rule

(3) Diff: $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$, $\frac{d}{dx}\log x = \frac{1}{x}$

Logarithms

$\log_b x$ inverse function to b^x.

point: $\log_b(xy) = \log_b x + \log_b y$

$\log_b(x^y) = y \cdot \log_b x$

(to compute $7.33 \cdot 10^8 \cdot 8.91 \cdot 10^6$, take logs:

$\log_{10}(7.33 \cdot 10^8) = 8 + \log_{10} 7.33$, $\log_{10}(8.91 \cdot 10^6) = 6 + \log_{10} 8.91$)

(mechanical implementation: "slide rule")

Examples: try $\log(e^{10}) = 10$ $\quad \log(2^{100}) = 100 \log 2$

Example: A variant on Moore's Law says the computing power roughly doubles every 18 months

(1) Suppose todays computer can do N_0 ops/second t years from now, computers will do

$$N(t) = N_0 \cdot 2^{t/1.5}$$

← in t years have $t/1.5$ doublings

(2) a computation will take 10 years today.
How long will it take if we wait 3 years?

It will take $\frac{10}{4} = 2.5$ years (two doublings in 3 years)
so will finish 5.5 years from now

(3) At what time will computers finish the task in 6 months?

After t years, computation takes $10/2^{t/1.5}$

so we want t s.t. $10 \cdot 2^{-t/1.5} = \frac{1}{2}$

i.e. $2^{t/1.5} = 20$ so $\frac{t}{1.5} \log 2 = \log 20$

so $\boxed{t = \frac{3}{2} \cdot \frac{\log 20}{\log 2}}$

Alternative: $2^{-t/1.5} = \frac{1}{20}$, so $-\frac{t}{1.5}\log 2 = \log\frac{1}{20}$

so $t = -1.5 \frac{\log \frac{1}{20}}{\log 2} = 1.5\frac{\log 20}{\log 2}$

$(\log_b \frac{1}{x} = -\log_b x)$

Source of errors: often can write Ne^{kt}, k negative

or Ne^{-kt}, k positive.

Logarithmic Diff

Last time: $\frac{d}{dx}(\log x) = \frac{1}{x}$. (if $x>0$)

if $x<0$, $\frac{d}{dx}(\log(-x)) = \frac{1}{-x}\cdot\frac{d(-x)}{dx} = \frac{-1}{-x} = \frac{1}{x}$

so: $\frac{d}{dx}\log|x| = \frac{1}{x}$ for $x \neq 0$

Examples: $\frac{d}{dx}\log(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}$

$\frac{d}{dt}\log(3t+t^2) = \frac{3+2t}{3t+t^2}$

or: $\log(3t+t^2) = \log t + \log(3+t)$

$\frac{d}{dx}(x^2\log(1+x^2)) =$

$\underset{\text{product rule}}{=} 2x\log(1+x^2) + x^2\frac{d}{dx}\log(1+x^2) =$

$= 2x\log(1+x^2) + x^2\cdot\frac{2x}{1+x^2}$

chain rule

or $\frac{d}{dx^2}(x^2\log(1+x^2)) \cdot 2x \cdot \frac{d(x^2\log(1+x^2))}{d(x^2)}$

$\frac{d}{dr}\frac{1}{\log(2+\sin r)}$

Example: diff $y=(x^2+1)\cdot\sin x\cdot\frac{1}{\sqrt{x^2+3}}\cdot e^{\cos x}$

Can do:

$$\log y = \log(x^2+1)+\log\sin x-\log\sqrt{x^2+3}+\cos x$$

so diff wrt x:

$$\frac{1}{y}\cdot y' = \frac{2x}{x^2+1}+\frac{\cos x}{\sin x}-\frac{1}{2}\,\frac{2x}{x^2+3}-\sin x$$

So $$y' = \underbrace{(x^2+1)\,\sin x\cdot\frac{1}{\sqrt{x^2+3}}\cdot e^{\cos x}}_{y}\cdot\left(\frac{2x}{x^2+1}+\frac{\cos x}{\sin x}-\frac{x}{x^2+3}-\sin x\right)$$

General rule: $\boxed{(\log f)' = \frac{f'}{f}} \Rightarrow \boxed{f' = f\cdot(\log f)'}$

(Bottom line: if it looks like $(\log f)'$ easier to compute, enough to compute it)

Example: diff x^x wrt x:

Also try $(\log x)^{\cos x}$

(2014 final) $y=x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x only.

if $y = x^x$, $\log y = \log(x^x) = x\log x$

so $\frac{d}{dx}(\log y) = \frac{d}{dx}(x\log x)$

i.e. $\frac{1}{y}\cdot\frac{dy}{dx} = \log x + \frac{x}{x} = \log x + 1$

so $\frac{dy}{dx} = (\log x + 1)\cdot x^x = x^x + x^x \log x$

Alt: $\frac{d}{dx}(x^x) = x^x \cdot \frac{d}{dx}(\log x^x) = x^x \cdot \frac{d}{dx}(x\log x) = x^x(\log x + 1)$

$\uparrow$
log diff
rule

Also, $x^x = e^{\log(x^x)} = e^{x\log x}$

(write f as e^g, so $f' = e^g g'$)

$\frac{d}{dx}\left((\log x)^{\cos x}\right) = (\log x)^{\cos x} \cdot \frac{d}{dx}(\cos x \log\log x) =$

$= (\log x)^{\cos x}\cdot\left(-\sin x \log\log x + \cos x \cdot \frac{1}{\log x}\cdot\frac{1}{x}\right)$

$= -\sin x \log\log x\, (\log x)^{\cos x} + \frac{\cos x}{x}\cdot(\log x)^{\cos x - 1}$

Question: If $y = (\log x)^{\cos x}$, is it $e^y = x^{\cos x}$?

$$e^y = e^{\left((\log x)^{\cos x}\right)} \neq \left(e^{\log x}\right)^{\cos x}$$

$$2^{(5^2)} = 2^{25} \qquad (2^5)^2 = 2^{5\cdot 2} = 2^{10}$$

true. $(\log x)^{\cos x} = e^{(\log\log x)\cdot \cos x}$

$$\log x = e^{\log\log x}.$$

Example: $y = x^{\log x}$. Then $\frac{dy}{dx} = x^{\log x} \cdot \frac{d}{dx}\left(\log\left(x^{\log x}\right)\right)$

$$= x^{\log x} \cdot \frac{d}{dx}(\log x \cdot \log x) = x^{\log x}\left(2\log x \cdot \frac{1}{x}\right)$$

$$= 2\log x \cdot x^{\log x - 1}$$

Question: Can't we say $\left(x^{\log x}\right)' = \log x \cdot x^{\log x - 1} \cdot (\log x)'$

chain rule.

Ex: Let $y = f^g$ where f, g depend on x.

Find $\frac{dy}{dx}$ in terms of f, g, f', g'.

Today: ~~saw~~ reviewed log, and log laws

diff: $\frac{d}{dx} \log|x| = \frac{1}{x}$

log diff: to diff $f \cdot g$, f^g can diff

$$\log(f \cdot g) = \cancel{\log} f + \log g$$
$$\log(f^g) = g \log f$$

instead

Equiv: if $y = f^g$ $\log y = g \log f$

diff use chain rule (for $\log y$)

Next time: use chain rule for ~~ma~~ more complicated expressions.

suppose we ~~consider~~ have the curve

$$\log(x+y) = x^2 + 7y$$

Can diff along the curve to get.

$$\frac{1}{x+y} \cdot (1+y') = 2x + 7y'$$