Math 100 - lecture 7, 26/9/2019

Plan: (1) Trig functions — values / differentiation

(2) The Chain Rule

~~Last time~~ Questions (from WW): we have a formula for $f(x+y)$. How do we compute $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$?

Hint: The formula also works for $f(x+h)$

Similar point:
"differentiate $\frac{1}{x+5}$": can also compute

$$\lim_{h\to 0} \frac{\frac{1}{t+5+h} - \frac{1}{t+5}}{h}$$

$f(*) =$ take $*$, add 5, compute inverse so $f(10) = \frac{1}{10+5}$

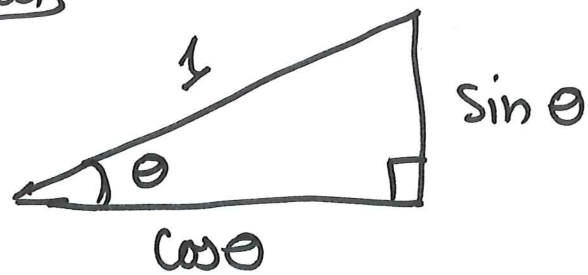
$f(x) = \frac{1}{x+5}$

$f(t) = \frac{1}{t+5}$

Last time: $\frac{d}{dx} x^r = r x^{r-1}$, $\frac{d}{dx} a^x = (\log a) \cdot a^x$, rules for algebraic combos of functions

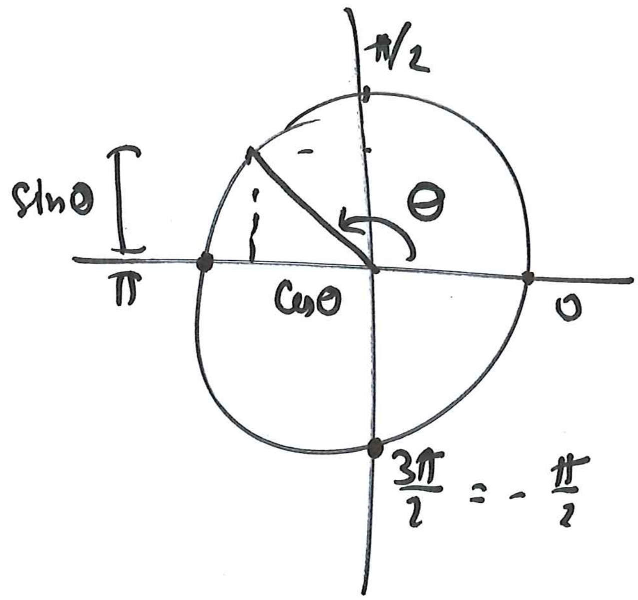
Trig functions

recall:



, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(only makes sense for $0 \le \theta \le \frac{\pi}{2}$, in general.



)

Fact: $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. This is $\lim_{h\to 0} \dfrac{\sin(\overset{0+h}{h}) - \sin(0)}{h} = 1$

i.e. $f(x) = \sin x$ is diff at $x=0$, and $\dfrac{d}{dx}\Big|_{x=0} (\sin x) = 1$

Use formulas like $\sin(x+h) = \sin x \cos h + \cos x \sin h$
to compute $\lim_{h\to 0} \dfrac{\sin(x+h) - \sin x}{h}$. Get: $\cos x$

similarly get $(\cos x)' = -\sin x$

Worksheet Section 1

Math 100 – WORKSHEET 7
TRIGONOMETRIC FUNCTIONS; THE CHAIN RULE

1. Trigonometric Functions

(1) (Special values) What is $\sin\frac{\pi}{3}$? What is $\cos\frac{5\pi}{2}$?

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{5\pi}{2}\right) = \cos\left(2\pi + \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

(2) Derivatives of trig functions

(a) Interpret $\lim_{h\to 0}\frac{\sin h}{h}$ as a derivative and find its value.

(b) Differentiate $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

By quotient rule, $\frac{d\tan\theta}{d\theta} = \frac{\cos\theta\cdot\cos\theta - \sin\theta\,(-\sin\theta)}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} =$

$= \frac{1}{\cos^2\theta}$ / Or: $\frac{d\tan\theta}{d\theta} = 1 + \tan^2\theta$

Date: 26/9/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(Also $\sec^2\theta$, where $\sec\theta = \frac{1}{\cos\theta}$)

The Chain Rule

$f'(x)$ is a rate of change: if we change x to $x+h$, $f(x)$ changes by (about) $f'(x)\cdot h$

What if f is a composition of effects?

Example: Suppose we consider the function

$$g(x) = f(10x)$$

if we change x by h, $10x$ is changes by $10h$

so $f(10x)$ is changed by

$$f'(10x)\cdot 10h$$

Summary: If we change x by h, $g(x)$ is changes by

$$(f'(10x)\cdot 10)\cdot h$$

so $g'(x) = f'(10x)\cdot 10$

Chain rule: If a function is a composition of other functions, derivatives multiply

(c) What is the equation of the line tangent the graph $y = T\sin x + \cos x$ at the point where $x = \frac{\pi}{4}$? Here T is a parameter (=constant).

$y\left(\frac{\pi}{4}\right) = T\sin\frac{\pi}{4} + \cos\frac{\pi}{4} = (T+1)\frac{\sqrt{2}}{2}.$

$\frac{dy}{dx} = T\cos x - \sin x$, so $\frac{dy}{dx}\left(\frac{\pi}{4}\right) = (T-1)\frac{\sqrt{2}}{2}.$

so line is .. $y = (T-1)\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{T+1}{\sqrt{2}}$

2. The Chain Rule

(1) Write the function as a composition and then differentiate.

(a) e^{3x}

$$\frac{d}{dx}(e^{3x}) = 3 \cdot \frac{d(e^{3x})}{d(3x)} = 3e^{3x}$$

(b) $\sqrt{2x+1}$

let $u = 2x+1$ then $\frac{du}{dx} = 2$, $\frac{d(\sqrt{u})}{du} = \frac{1}{2\sqrt{u}}$

so $\frac{d(\sqrt{u})}{dx} = 2 \cdot \frac{1}{2\sqrt{u}} = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{2x+1}}$

changing x causes u to change

changing u causes $\sqrt{u}$ to change

let $f(u) = \sqrt{u}$

$g(x) = 2x+1$

then $f(g(x)) = \sqrt{2x+1}$

Two approches to $\frac{d}{dx} \sin(x^2)$

(1) Know $\frac{d(x^2)}{dx} = 2x$, so $\frac{d}{dx} \sin(x^2) = 2x \cdot \frac{d}{d(x^2)} \sin(x^2)$

$= 2x \cos(x^2)$

(2) Know to diff sin fcn:

$$\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot (\text{diff } x^2) = \cos(x^2) \cdot 2x$$

diff sin θ to get cos θ

thinking/ informal

(b) $e^{\sqrt{\cos x}}$

$$\frac{d}{dx} e^{\sqrt{\cos x}} = e^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot (-\sin x)$$

$$= -\frac{\sin x}{2\sqrt{\cos x}} \cdot e^{\sqrt{\cos x}}$$

optional

(c) (Final 2012) $e^{(\sin x)^2}$

$$\frac{d}{dx}\left(e^{(\sin x)^2}\right) = e^{(\sin x)^2} \cdot 2\sin x \cdot \cos x$$

(c) (Final, 2015) $\sin(x^2)$

Let $g(x) = x^2$, $f(u) = \sin u$. Then $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

$= \cos(x^2) \cdot 2x.$

(d) $(7x + \cos x)^n$. (n fixed)

Let $u = 7x + \cos x$, then $(7x + \cos x)^n = u^n$.

So $\frac{d(u^n)}{dx} = \frac{d(u^n)}{du} \cdot \frac{du}{dx} = nu^{n-1}(7 - \sin x)$

$= n(7x + \cos x)^{n-1} \cdot (7 - \sin x)$

(2) Differentiate

(a) $7x + \cos(x^n)$

$\frac{d}{dx}(7x + \cos(x^n)) = \frac{d}{dx}(7x) + \frac{d}{dx}(\cos(x^n)) = 7 - \sin(x^n) \cdot nx^{n-1}$

linearity