6. Polynomials and Exponentials (24/9/2019)

Goals.

(1) Exponentials: laws of powers
(2) Derivatives of exponentials
(3) The tangent line

Last Time.

Diff rules: $(f\pm g)' = f'\pm g'$, $(f\cdot g)' = f'g + fg'$, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

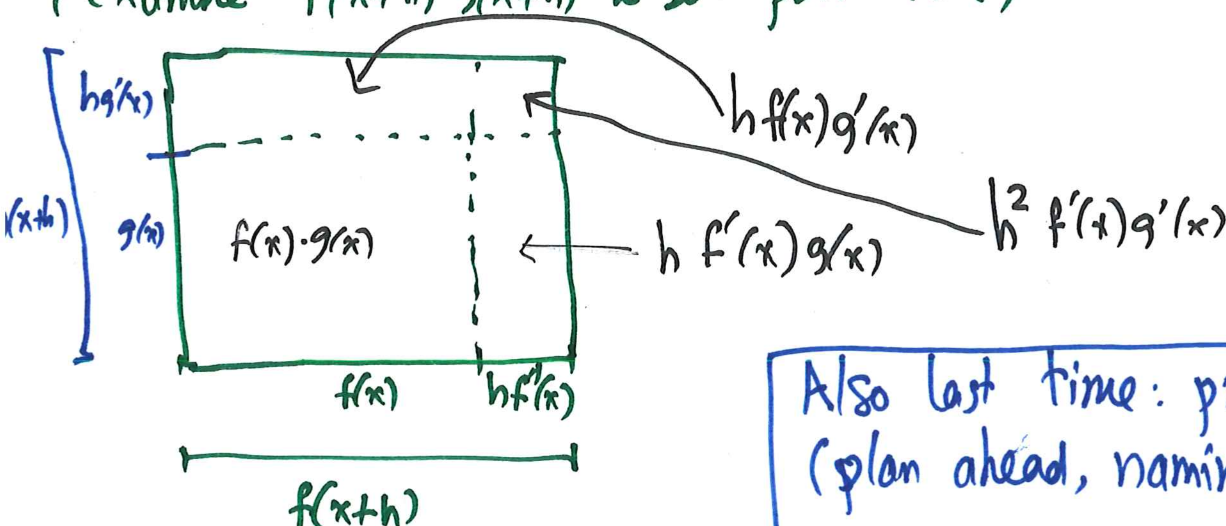
$\frac{d}{dx}(x^r) = r x^{r-1}$

Why isn't the product rule $(fg)' = f'g'$?

New pov: instead of $f'(x) \approx \frac{f(x+h)-f(x)}{h}$, think

$$f(x+h) \approx f(x) + h f'(x).$$

(Examine $f(x+h)\cdot g(x+h)$ to get pdt rule)



Also last time: problem-solving (plan ahead, naming quantities)

Math 100 – WORKSHEET 6
POLYNOMIALS AND EXPONENTIALS

1. Direct problems

(1) Differentiate

(a) $f(x) = 6x^{\pi} + 2x^{e} - x^{7/2}$

$f'(x) = 6\pi x^{\pi-1} + 2e x^{e-1} - \frac{7}{2} x^{5/2}$

(using power law rule, linearity)

(b) (Final, 2016) $f(x) = x^2 e^x$ (also try $x^a e^x$)

By product rule, $(x^2 e^x)' = 2x \cdot e^x + x^2 e^x = x(x+2)e^x$

$(x^a e^x) = a x^{a-1} e^x + x^a e^x = x^{a-1}(x+a)e^x$

(c) (Final, 2016) $f(x) = \frac{x^2+3}{2x-1}$

By quotient rule, $f'(x) = \frac{2x \cdot (2x-1) - (x^2+3) \cdot 2}{(2x-1)^2} = \frac{2x^2 - 2x - 6}{(2x-1)^2} =$

$= 2\frac{x^2 - x - 3}{(2x-1)^2}$

Date: 24/9/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Exponential functions

Laws of exponentials:
$$\begin{cases} q^x q^y = q^{x+y} \\ q^x \cdot r^x = (qr)^x \\ (q^x)^y = q^{xy} \end{cases}$$

Warning: $\left((-8)^{1/3}\right)^2 = -2^2 = 4$

$\left((-8)^2\right)^{1/3} = 64^{1/3} = 4$

Exercise: want $\frac{d}{dx}(7^x) = \lim_{h\to 0} \frac{7^{x+h} - 7^x}{h} = \lim_{h\to 0} \frac{7^x 7^h - 7^x}{h} =$

$$= \lim_{h\to 0} 7^x \cdot \frac{7^h - 1}{h} = 7^x \lim_{h\to 0} \frac{7^h - 1}{h}$$

7^x indep of h

call $\lim_{h\to 0} \frac{7^h-1}{h} = L(7)$

get $\frac{d}{dx}(7^x) = L(7)\cdot 7^x$

in general, for any base $b>0$, have $L(b)$ s.t.

$$\frac{d}{dx}(b^x) = L(b)\cdot b^x.$$

Facts: $L(2) \approx 0.693\ldots$

$L(3) \approx 1.098\ldots$

There is e, $2<e<3$ s.t. $L(e)=1$, then $(e^x)'=e^x$.

(it turns out that $e \approx 2.71828\ldots$

$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$)

In fact, $L(a)$ is the natural logarithm of a.

So $\frac{d}{dx}(7^x) = (\log 7) \cdot 7^x$.

Aside on notation: Every field uses log to mean "logarithm to the most common basis".

Engineering: log usually means $\log_{10}$

CS: " " " $\log_2$

Math: " " " $\log_e = \ln$

Worksheet §2

(d) $f(x) = \frac{\sqrt{x}(1-3x)}{x^2+1}$

note: $\left(\sqrt{x}(1-3x)\right)' = \frac{1}{2\sqrt{x}}(1-3x) - 3\sqrt{x}$

Also

$= \left(\sqrt{x} - 3x^{3/2}\right)' = \frac{1}{2\sqrt{x}} - \frac{9}{2}x^{1/2}$

(e) $f(x) = \frac{x^2+xe^x}{\cos x+\sin x}$

2. Exponentials

(1) Simplify $\left(e^5\right)^3$, $\left(2^{1/3}\right)^{12}$, 7^{3-5}. $7^3 \cdot 7^{-5}$

$(e^5)^3 = e^{5\cdot 3} = e^{15}$, $(\sqrt[3]{2})^{12} = 2^{\frac{1}{3}\cdot 12} = 2^4 = 16$, $7^3 \cdot 7^{-5} = 7^{3-5} = 7^{-2} = \frac{1}{49}$

(2) Differentiate:

base fixed

exponent variable

(a) 10^x

$\frac{d}{dx}(10^x) = (\log 10) \cdot 10^x$

$\boxed{\text{not } x10^{x-1}}$

(b) $\frac{5\cdot 10^x+x^2}{3^x+1}$

$\frac{d}{dx}\left(\frac{5\cdot 10^x + x^2}{3^x+1}\right) = \frac{(5\log 10\cdot 10^x + 2x)\cdot(3^x+1) - (5\cdot 10^x + x^2)\log 3\cdot 3^x}{(3^x+1)^2}$

Ex: use prdt rule to show

$L(ab) = L(a) + L(b)$

A: $\frac{d}{dx}(a^x b^x) = \left(\frac{d}{dx}a^x\right)\cdot b^x + a^x \cdot \frac{d}{dx}(b^x)$

$\frac{d}{dx}((ab)^x)$

Question: what about $\frac{d}{dx}(x^x)$ A: wait for class on "logarithmic diff."

(or write $x^x = e^{x \log x}$, use chain rule)

Tangent lines

Recall: If $f(x)$ is diff at $x=a$, then the line tangent to the graph $y=f(x)$ at $(a, f(a))$ is the line with equation

$$y - f(a) = f'(a)\cdot(x-a)$$

or

$$y = f'(a)(x-a) + f(a)$$

or

$$y = f'(a)x + f(a) - f'(a)a$$

Useful: (1) given f, can find line (& approximate f using this line)

(2) given line, can find $f(a)$, $f'(a)$

3. Tangent lines

(1) Suppose that $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, $g'(1) = 4$. Find $(fg)'(1)$ and $\left(\frac{f}{g}\right)'(1)$.

$(fg)'(1) = f'(1)\cdot g(1) + f(1)g'(1) = 3\cdot 2 + 1\cdot 4 = 10$

$\left(\frac{f}{g}\right)'(1) = \dfrac{f'(1)g(1) - f(1)g'(1)}{g(1)^2} = \dfrac{3\cdot 2 - 1\cdot 4}{2^2} = \dfrac{1}{2}$

(2) (Final, 2015) Find the equation of the line tangent to the function $f(x) = \sqrt{x}$ at $(4, 2)$.

The answer is not $y = \frac{1}{2\sqrt{x}}(x-4)+2$

error

The answer is $y = \frac{1}{2\sqrt{4}}(x-4) + 2 = \frac{1}{4}(x-4) + 2$

(3) Find the lines of slope 3 tangent the curve $y = x^3 + 4x^2 - 8x + 3$.

Let x be the x-co-ord of the point of tangency of such a line.

Then $\left.\frac{dy}{dx}\right|_x = 3$, i.e. $3x^2 + 8x - 8 = 3$

so $3(x^2-1) + 8(x-1) = 0$, $\Leftrightarrow (x-1)(3(x+1)+8) = 0$

$\Leftrightarrow (x-1)(3x+11) = 0$ so $x=1$ or $x = -\frac{11}{3}$.

so the lines are $y = 3(x-1)$, and $y = 3\left(x + \frac{11}{3}\right) + \left(\left(\frac{11}{3}\right)^3 + 4\left(\frac{11}{3}\right)^2 - 8\cdot\frac{11}{3} + 3\right)$

(4) Let $f(x) = \frac{g(x)}{x}$, where $g(x)$ is differentiable at $x = 1$. The line $y = 2x - 1$ is tangent to the graph $y = f(x)$ at $x = 1$. Find $g(1)$ and $g'(1)$.

The line has slope 2, passes through (1,1).

So $f(1) = 1$, $f'(1) = 2$ Thus $1 = f(1) = \frac{g(1)}{1} = g(1)$, $\boxed{g(1) = 1}$

$$2 = f'(1) = \frac{g'(1) \cdot 1 - g(1)}{1^2} = g'(1) - 1 \quad \text{so} \quad \boxed{g'(1) = 3}$$

$$f'(x) = \frac{g'(x) \cdot x - g(x)}{x^2}$$

(5) (Final 2015) The line $y = 4x + 2$ is tangent at $x = 1$ to which function: $x^3 + 2x^2 + 3x$, $x^2 + 3x + 2$, $2\sqrt{x+3} + 2$, $x^3 + x^2 - x$, $x^3 + x + 2$, none of the above?

Hint: the line has slope 4, passes through (1, 6)

first see which functions have $f(1) = 6$