Math 100, lecture 3, 12/9/2019

(0) Squeeze thm

(1) limits at ∞ — "gut" method / formal calculation

(2) Continuity

Last time: (1) calculation of limits, limit laws.
(limits behave as you expect)

Example: If $f(x) \xrightarrow[x\to 3]{} 5$ then $\sqrt{f(x)+7} \xrightarrow[x\to 3]{} \sqrt{5+7} = \sqrt{12}$

(difficulty: no formula for f)

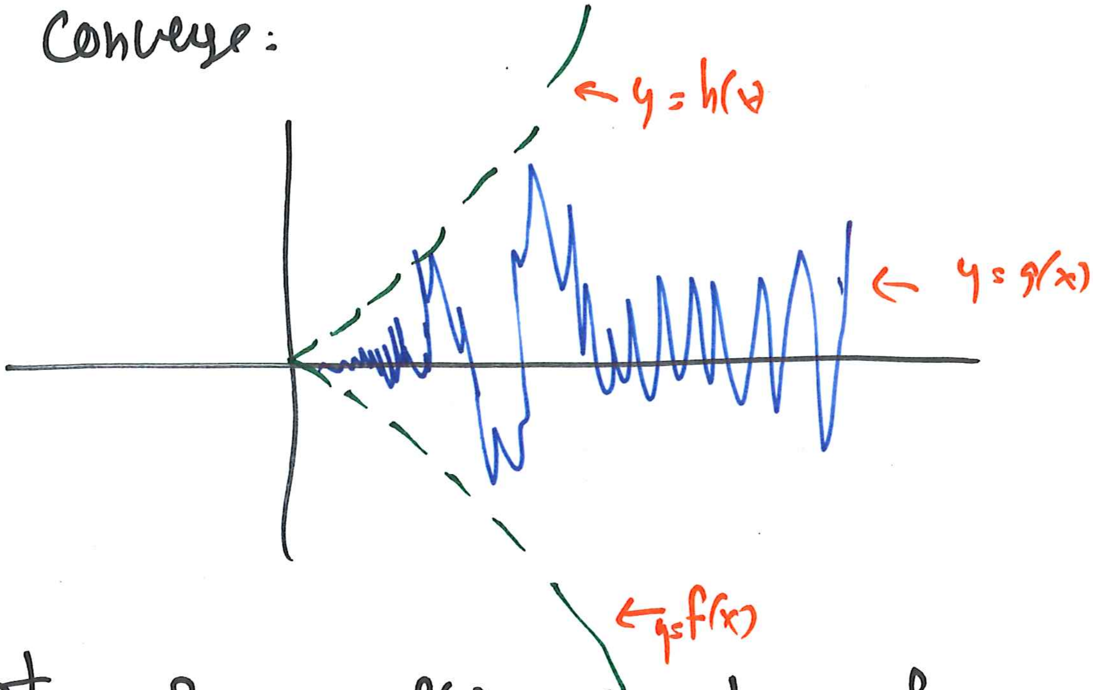
(2) trick: $\sqrt{a} - \sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}$

Note: consider $a+b$. If a much bigger than b, $a+b \approx a$. If b much bigger, $a+b \approx b$.
If a, b are of same scale, sum of same scale.

Differences are different. a, b can be big with $a-b$ small

The squeeze thm

<u>Idea:</u> Two "guideline" can force a function to converge:



<u>Thm:</u> Suppose $f(x) \le g(x) \le h(x)$ for x near x_0

And suppose $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = A$

<u>Then</u> $\lim_{x\to x_0} g(x) = A$ as well.

(also works for one-sided limits)

<u>Example:</u> Find $\lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right)$

<u>Wrong:</u> $\lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right) = \left(\lim_{x\to 0^+} x\right)\left(\lim_{x\to 0^+} \cos\left(\frac{1}{x}\right)\right) = 0\cdot\left(\lim_{x\to 0^+} \cos\left(\frac{1}{x}\right)\right) = 0$

(consider $\lim_{x\to 0^+} x\cdot\frac{1}{x} \neq 0$)

<u>Correct:</u> for any x, $-1 \le \cos\left(\frac{1}{x}\right) \le 1$ so if $x > 0$

$$-x \le x\cos\left(\tfrac{1}{x}\right) \le x$$

Also, $\lim_{x\to 0^+}(-x) = 0 = \lim_{x\to 0^+} x$. By the squeeze thm, $\lim_{x\to 0^+} x\cos\left(\frac{1}{x}\right) = 0$

Math 100 – WORKSHEET 3
LIMITS AT INFINITY; CONTINUITY

1. The Squeeze Theorem

(1) $\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right)$.

For all $x \neq 0$, $-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$ and $x^2 > 0$

so $-x^2 \leq x^2 \sin\left(\frac{\pi}{x}\right) \leq x^2$

Also, $\lim_{x\to 0} x^2 = 0$ so $\lim_{x\to 0} -x^2 = -0 = 0$ and, by the squeeze thm,

$\lim_{x\to 0} x^2 \sin\left(\frac{\pi}{x}\right) = 0$ as well.

(2) (Final, 2014) Suppose that $8x \leq f(x) \leq x^2 + 16$ for all $x \geq 0$. Find $\lim_{x\to 4} f(x)$.

Date: 12/9/2019, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

~~Limits at infinity~~

Examples consider $\lim_{x \to 0} \left(1 + x^2 \sin\left(\frac{\pi}{x}\right)\right)$

Limits at ∞

So far we asked "what happens to $f(x)$ if x is really close to x_0?"

New question: What happens when x is really large

Examples $\lim_{x \to \infty} \frac{1}{x} = 0$, $\lim_{x \to \infty} x^{-a} = 0$, $a > 0$

$\lim_{x \to \infty} e^{-x} = 0$

But $\lim_{x \to \infty} \cos x$ DNE.

$\lim_{x \to \infty} x = \infty$ (DNE, but ∞ in extended sense)

Examples $\lim_{x \to \infty} \frac{x+1}{x+2} =$

informal: $\frac{x+1}{x+2} \sim \frac{x}{x} \underset{x \to \infty}{\to} 1$

$\lim_{x \to \infty} \frac{x^2+1}{x+2} =$

informal: $\frac{x^2+1}{x+2} \sim \frac{x^2}{x} = x \underset{x \to \infty}{\to} \infty$

$\lim_{x \to \infty} \frac{x+1}{x^2+2} =$

informal: $\frac{x+1}{x^2+2} \sim \frac{x}{x^2} = \frac{1}{x} \underset{x \to \infty}{\to} 0$

2. Limits at infinity

(1) Evaluate the following limits:

(a) $\lim_{x\to\infty} \frac{x^2+1}{x-3} = \lim_{x\to\infty} \frac{x^2(1+1/x^2)}{x(1-3/x)} = \lim_{x\to\infty} x \cdot \left(\frac{1+1/x^2}{1-3/x}\right) = \infty$

$x \to \infty$

$(\ ) \xrightarrow[x\to\infty]{} 1$

informally: $\frac{x^2+1}{x-3} \sim \frac{x^2}{x} \sim x \to \infty$

(b) (Final, 2015) $\lim_{x\to\infty} \frac{x+1}{x^2+2x-8} =$

$\frac{x+1}{x^2+2x+8} \sim \frac{x}{x^2} \sim \frac{1}{x} \to 0$

$\frac{x+1}{x^2+2x-8} = \frac{(1+\frac{1}{x})x}{(1+\frac{3}{x}-\frac{8}{x^2})x^2} = \frac{1}{x} \cdot \frac{1+\frac{1}{x}}{1+\frac{2}{x}-8/x^2} \xrightarrow[x\to\infty]{} 0 \cdot \frac{1+0}{1+0+0} = 0$

(c) (Quiz, 2015) $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} =$

informally: $\frac{3x}{\sqrt{4x^2+x}-2x} \sim \frac{3x}{\sqrt{4x^2}-2x} \sim \frac{3x}{-2x-2x} = -\frac{3x}{4x} \to -\frac{3}{4}$

$x<0$

Formally: $4x^2+x = (4x^2)(1+\frac{1}{4x})$

So $\sqrt{4x^2+x} = \sqrt{4x^2}\sqrt{1+\frac{1}{4x}}$

$= (-2x)\sqrt{1+\frac{1}{4x}}$

$x<0$

(d) $\lim_{x\to\infty} \frac{\sqrt{x^4+\sin x}}{x^2-\cos x} =$

$\frac{\sqrt{x^4+\sin x}}{x^2-\cos x} = \frac{\sqrt{x^4(1+\frac{\sin x}{x^4})}}{x^2(1-\frac{\cos x}{x^2})}$

(e) $\lim_{x\to-\infty} \left(\sqrt{x^2+2x} - \sqrt{x^2-1}\right) =$

So $\lim_{x\to-\infty} \frac{3x}{\sqrt{4x^2+x}-2x} = \lim_{x\to-\infty} \frac{3x}{(-2x)\sqrt{1+\frac{1}{4x}}-2x} = \lim_{x\to-\infty} \frac{3}{-2\sqrt{1+\frac{1}{4x}}-2}$

$= \frac{3}{-2\sqrt{1+0}-2} = -\frac{3}{4}.$

Formally: Extract rate of growth from function

$$x+1 = x\left(1+\frac{1}{x}\right), \qquad x+2 = x\left(1+\frac{2}{x}\right)$$

$$x^2+1 = x^2\left(1+\frac{1}{x^2}\right), \qquad x^2+2 = x^2\left(1+\frac{2}{x^2}\right)$$

$$\lim_{x\to\infty} \frac{x+1}{x+2} = \lim_{x\to\infty} \frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\left(1+\frac{2}{x}\right)} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1+\frac{2}{x}} = \frac{1+0}{1+0} = 1$$

Worksheet (1) a, b

Note: $\sqrt{x^2} = |x| = \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$

So if x is very negative, $\sqrt{4x^2+x} \approx -2x$

$$\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty$$

$$\lim_{x\to 0^-} e^{1/x} = \lim_{x\to 0^-} \frac{1}{e^{-\frac{1}{x}}} = 0$$

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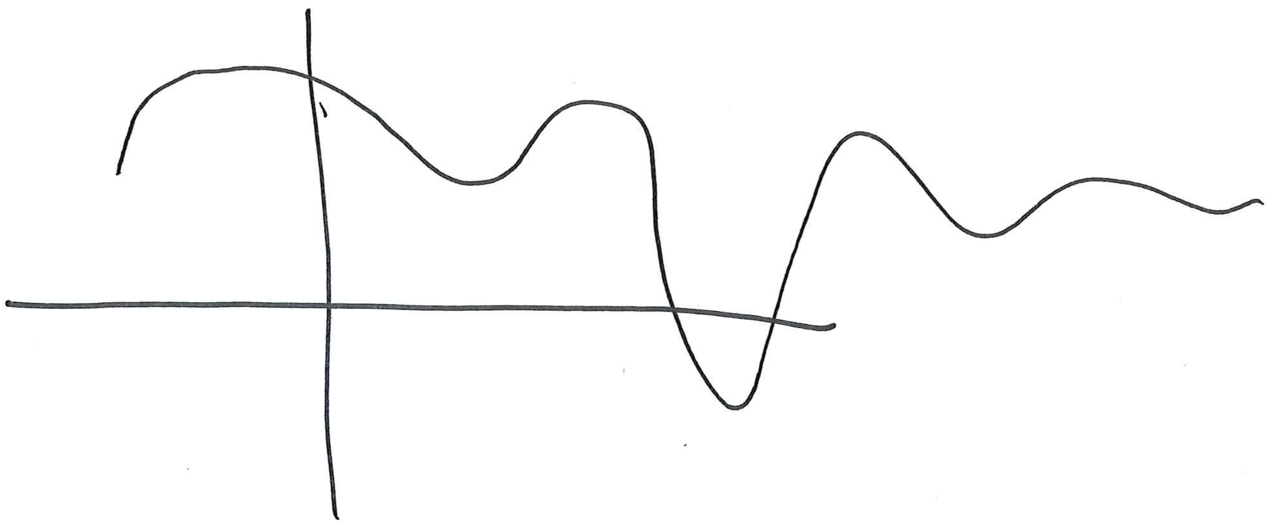
if x is negative, $\sqrt{x^2} = -x$

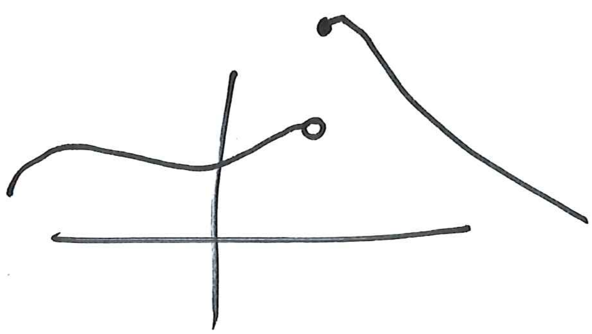
Continuity

Call f continuous (at a point, on an interval, on $\mathbb{R}$)
if at every relevant point x_0, $\lim_{x \to x_0} f(x)$ exists
and equals $f(x_0)$

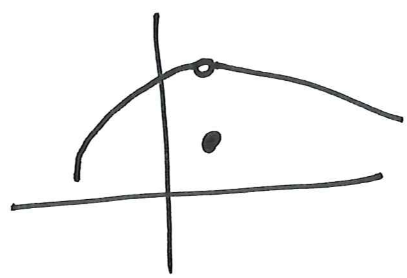


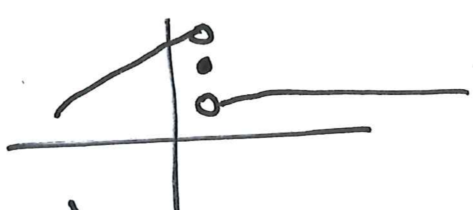
Examples of discontinuities:

(1) jump

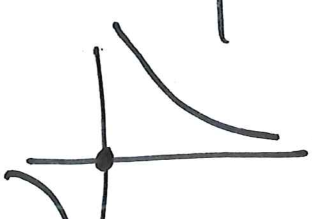


(2) "hole"



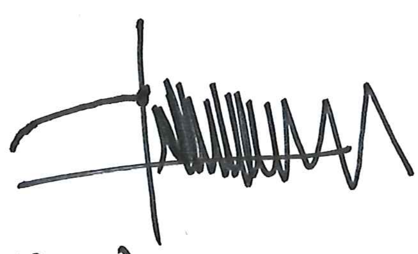


(3)



limits DNE

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$$f(x) = \begin{cases} 1 & x \leq 0 \\ \sin(\frac{1}{x}) & x > 0 \end{cases}$$

<u>Fact:</u> If f is defined by formula,
f is cts wherever the formula makes sense