Lior Silberman’s Math 535, Problem Set 4: Preliminaries on Tori

Connected abelian Lie groups

1. Let \( \Lambda < \mathbb{R}^d \) be a discrete subgroup. Show that \( \Lambda = \bigoplus_{i=1}^k \mathbb{Z}v_i \) for a linearly independent set \( \{v_i\}_{i=1}^k \subset \mathbb{R}^d \). Conversely show that such a subgroup is discrete.

2. Let \( G \) be an Abelian Lie group, and suppose that \( \pi_0(G) = G/\mathcal{G} \) is finite. Show that \( G \cong \mathcal{G} \times \pi_0(G) \). (Hint: show that a connected abelian Lie group is divisible).

Tori

3. (Fourier analysis on tori) Let \( T_n = \mathbb{R}^n / \mathbb{Z}^n \) be the \( n \)-torus. A trigonometric polynomial on \( T_n \) is a function of the form \( f(x) = \sum_{k=1}^l a_k e(k \cdot x) \) where \( k \in (\mathbb{Z}^n)^* \) lie in the dual lattice.

   (a) Use Peter–Weyl to show that the space of trigonometric polynomials is dense in \( C(T_n) \) and \( L^2(T_n) \).

   (b) Use Stone–Weierstrass instead to show that the trigonometric polynomials are dense in \( C(T_n) \), and use that to show that their orthocomplement in \( L^2(T_n) \) vanishes, getting density there too.

   (c) For \( f \in L^2(T_n) \) and \( k \in (\mathbb{Z}^n)^* \) set \( \hat{f}(k) = \int_{T_n} f(x)e(-k \cdot x) \, dx \) (probability Haar measure). Then \( \sum_k \hat{f}(k)e(k \cdot x) \) converges in \( L^2 \) to \( f \).

   (d) For \( f \in C^m(T_n) \) use integration by parts to show that \( \left| \hat{f}(k) \right| \leq C_f \left( 1 + |k| \right)^{-m} \). Conclude that for \( m > n \), the series \( \sum_k \hat{f}(k)e(k \cdot x) \) converges in \( C^{m-n-1} \) to \( f \).

   (e) (Weyl criterion) Let \( \{\mu_j\}_{j=1}^\infty \) be a sequence of Borel probability measures on \( T_n \). Show that \( \mu_j(f) \to \mu(f) \) for every \( f \) iff this holds for the plane waves \( f(x) = e(k \cdot x) \).

4. (Weyl equidistribution) Let \( \{\xi_i\}_{i=1}^n \subset \mathbb{R} \) be linearly independent over \( \mathbb{Q} \) where \( \theta_0 = 1 \), and let \( \xi = (\xi_i)_{i=1}^n \mod \mathbb{Z}^n \in T_n \). Show that the sequence \( \left\{ k\xi \right\}_{k=1}^\infty \subset T_n \) is uniformly distributed: for any open \( U \subset T_n \),

\[
\frac{1}{K} \# \left\{ 1 \leq k \leq K \mid k\xi \in U \right\} = \frac{\text{vol}(U)}{\text{vol}(T_n)}.
\]

Conclude that the sequence \( \left\{ k\xi \right\}_{k=1}^\infty \) is dense in the torus.

Hint: Let \( \mu_K = \frac{1}{K} \sum_{k=1}^K \delta_{k\xi} \). By 1(e) to show \( \mu_K \xrightarrow{\text{wk-*}} \text{vol} \) it suffices to test against plane waves.