Lior Silberman’s Math 535, Problem Set 3: Lie Groups

Constructions on manifolds

The first two exercises are highly recommended if you are interested in algebraic geometry or in differential geometry:

1. (View of the tangent space) Let $M$ be a smooth manifold, $\mathcal{C}^\infty(M)$ its algebra of smooth functions (multiplication defined pointwise). For $p \in M$ let $I_p = \{ f \in \mathcal{C}^\infty(M) \mid f(p) = 0 \}$ be the associated maximal ideal. Recall that we set $T^*_pM = I_p/I^2_p$ and $T_pM = (T^*_pM)^\vee$.
   (a) Let $G_p$ be the set of pairs $(f,U)$ where $p \in U \subset M$ is open and $f \in \mathcal{C}^\infty(U)$. Show that $(f,U) \sim (g,V) \iff f \mid_{U \cap V} = g \mid_{U \cap V}$ is an equivalence relation, and endow $\mathcal{G}_p \coloneqq G_p/\sim$ with a natural structure as an $\mathbb{R}$-algebra.
   (b) Let $\mathcal{C}^\infty(M)_p$ be the localization of $\mathcal{C}^\infty(M)$ at the prime ideal $I_p$. Show that associating to $f \in \mathcal{C}^\infty(M)$ the equivalence class of $(f,M) \in G_p$ is an algebra homomorphism $\mathcal{C}^\infty(M) \to \mathcal{G}_p$ inducing an isomorphism $\mathcal{C}^\infty(M)_p \simeq \mathcal{G}_p$.
   (c) Conclude that restriction of maps induces an isomorphism $I_p(M)/I^2_p(M) \simeq I_p(U)/I^2_p(U)$ for any open $U$ containing $p$.
   (d) A derivation at $p$ is an $\mathbb{R}$-linear map $X : \mathcal{C}^\infty(M) \to \mathbb{R}$ such that $X(fg) = (Xf)g(p) + (Xg)f(p)$. Write $\tilde{T}_pM$ for the set of derivations at $p$. Show that $\tilde{T}_pM$ is an $\mathbb{R}$-vector space.
   (e) Conversely, let $v \in T_pM$. Show that setting $X_v f \coloneqq v(f - f(p))$ gives $X_v \in \tilde{T}_pM$ and that the map $v \mapsto X_v$ is inverse to the map of (d).

2. Let $M,N$ be smooth manifolds and let $\varphi : M \to N$ be a smooth map. Fix $p \in M$.
   (a) Show that mapping $f \in I_{\varphi(p)}N$ to $f \circ \varphi \in I_p(M)$ induces a linear map $(d\varphi_p) : T^*_{\varphi(p)}N \to T^*_pM$.
   (b) For $X \in \tilde{T}_pM$ and $f \in \mathcal{C}^\infty(N)$ set $d\varphi_p(X)f \coloneqq X(f \circ \varphi)$. Show that $d\varphi_p(X) \in \tilde{T}_pN$ and that $d\varphi_p \in \text{Hom}_\mathbb{R}(\tilde{T}_pM, \tilde{T}_pN)$.
   (c) Show that, under the isomorphism $T_p$ and $\tilde{T}_p$ from problem 1, the maps $d\varphi_p$ and $(d\varphi_p)^*$ are indeed dual.
   (d) Show that the map $\varphi \mapsto d\varphi$ satisfies the chain rule: if $\psi : L \to M$ is smooth and $p \in L$ then $d(\varphi \circ \psi)_p = d\varphi_{\psi(p)} \circ d\psi_p$.
   (e) Show that $d\varphi_p, (d\varphi_p)^*$ extend to bundle maps $d\varphi : TM \to TN, d\varphi^* : T^*N \to T^*M$. 

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The following two exercises are merely a technical verification.

**Definition.** Let $\Omega \subset \mathbb{R}^n$ be a domain, $V$ a topological vector space. For $1 \leq i \leq n$, $f : \Omega \to V$ and $x \in \mathbb{R}^n$ set
\[
(\partial_i f)(x) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}
\]
($e_i$ is the unit vector in direction $i$) provided the limit exists. Write $C^0(\Omega; V) = C(\Omega; V)$ for the space of continuous functions $\Omega \to V$ and then let
\[
C^{k+1}(\Omega; V) = \left\{ f \in C^k(\Omega; V) \mid \forall i : \partial_i f \in C^k(\Omega; V) \right\}
\]
\[
C^\infty(\Omega; V) = \bigcap_{k=0}^\infty C^k(\Omega; V).
\]
Finally, if $V$ is a normed space we set $\|f\|_{C^k} = \sup \{ \|\partial^\alpha f(x)\| \mid x \in \Omega, |\alpha| \leq k \}$.

3. Show that this definition is independent of the choice of co-ordinates: if $\varphi : \Omega \to \Omega'$ is a diffeomorphism then $f \mapsto f \circ \varphi$ is a bijection $C^k(\Omega'; V) \to C^k(\Omega; V)$. In particular, $f \in C^1(\Omega; V)$ has directional derivatives in all directions.

4. Let $M$ be a smooth manifold. Define the spaces $C^k(M; V)$ and $C^\infty(M; V)$. Show that, provided $M$ is compact, $\|f\|_{C^k} < \infty$ for all $f \in C^k(M; V)$.

**Representation Theory**

Fix a Lie group $G$ and a representation $(\pi, V) \in \text{Rep}(G)$.

5. Call $v \in V$ smooth if the orbit function $g \mapsto \pi(g)v$ is a smooth function $G \to V$. Write $V^\infty$ for the set of smooth vectors in $V$.
   (a) Show that $V^\infty$ is a $G$-invariant subspace of $V$.
   (b) Show that $V^\infty$ is dense in $V$ (hint: revisit arguments used in the proof of the Peter–Weyl Theorem).
   (c) Suppose $V$ is finite dimensional. Show that the homomorphism $\pi : G \to \text{GL}(V)$ is a smooth map of smooth manifolds.

6. For $X \in \mathfrak{g}$ and $v \in V^\infty$ set $\pi(X)v = \frac{d}{dt} \bigg|_{t=0} \pi(e^{tX})v$.
   (a) Show that this is well-defined (that the derivative above exists) and that $\pi(X)v \in V^\infty$. In fact, show that $\pi(X) : V^\infty \to V^\infty$ is linear.
   (b) (Compatibility) Show that $\pi(g)\pi(X)\pi(g^{-1}) = \pi(\text{Ad}_g X)$ for all $g \in G$.
   (c) Show that $X \mapsto \pi(X)$ is a linear map $\mathfrak{g} \to \text{End}_C(V^\infty)$.
   (d) Show that we have a Lie algebra representation: $\pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X) = [\pi(X),\pi(Y)]$. Here, the first commutator is the one in $\mathfrak{g}$, the second the one of $\text{End}_C(V^\infty)$.

**Structure theory**

7. Show that $\exp : \mathbb{R}^n \to \text{SL}_n(\mathbb{R})$ is not surjective.