Lior Silberman’s Math 535, Problem Set 1b: Analysis

Haar measure

Let $X$ be a locally compact topological space. Write $C(X)$ for the space of continuous real-valued functions on $X$, and for $f \in C(X)$ write $\|f\|_{\infty} = \sup \{|f(x)| : x \in X\}$. It is well-known that the subspace $C_b(X) = \{ f \in C(X) : \|f\|_{\infty} < \infty \}$ is complete in the supremum norm and that it contains the subspace $C_c(X)$ of compactly supported functions.

**Definition.** A Radon measure on $X$ is a linear functional $\mu : C_c(X) \to \mathbb{C}$ such that $\mu(f) \geq 0$ if $f \geq 0$ (that is, if $f(x) \in \mathbb{R}_{\geq 0}$ for each $x$). If $\mu$ is a Radon measure and $f \in C_c(X)$ we often write $\int f \, d\mu$ instead of $\mu(f)$.

1. **(Preliminaires)**
   (a) Show that the closure of $C_c(X)$ in $C_b(x)$ is the space $C_0(X)$ of functions vanishing at infinity (continuous functions $f$ such that for all $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ if $x \notin K$).
   (b) Let $X'$ be a Haar measure on $X$. Show that $\mu \upharpoonright_{C_c(X')}^\ast$ is a Radon measure on $X'$.
   (b) In particular, suppose $Y$ is compact. Show that a Radon measure on $Y$ is a bounded linear functional on $C(Y) = C_b(Y) = C_c(Y)$.

2. Let $G$ be a locally compact topological group.
   (a) Let $f, f' \in C_c(G)$ be non-negative, and let $U \subset G$ be open. Set
   $$\langle f : U \rangle = \inf \left\{ \sum_{i=1}^{n} \alpha_i \mid \alpha_i \geq 0, f \leq \sum_{i=1}^{n} \alpha_i : 1_{gU} \right\}.$$
   Show that $0 \leq \langle f : U \rangle < \infty$. Assuming $f' \neq 0$ show that $\langle f : U \rangle \leq \langle f' : U \rangle \langle f : f' \rangle$ for an appropriately defined $\langle f : f' \rangle$ which is independent of $U$.
   (b) Let $\mathcal{N}$ be the set of open neighbourhoods of the identity in $G$; for $U \in \mathcal{N}$ set $F_U = \{ V \in \mathcal{N} \mid V \subset U \}$. Show that $\mathcal{F} = \{ S \subset \mathcal{N} \mid \exists U : S \supset F_U \}$ is a filter on $\mathcal{N}$ (that is, if $S_1, S_2 \in \mathcal{F}$ and $T \subset \mathcal{N}$ then $S_1 \cap S_2, S_1 \cup T \in \mathcal{F}$). Show that for any $V \in \mathcal{N}$ there is $S \in \mathcal{F}$ with $V \not\subset S$ (“$\mathcal{F}$ is not contained in any principal filter”). Let $\omega \in \mathcal{N}$ be a maximal filter containing $\mathcal{F}$.
   (c) Fix $f_0 \in C_c(G)$ which is non-negative and non-zero. Show that $\mu(f) \overset{\text{def}}{=} \lim_{U \to \omega} \langle f : U \rangle \langle f : f_0 \rangle$ extends to a $G$-invariant Radon measure on $G$. Such $\mu$ is called a (left) **Haar measure** on $G$.
   (d) Show that $\mu(f) > 0$ for all non-negative non-zero $f \in C_c(G)$.
   (e) Suppose $G$ is non-compact. Show that $\mu$ is an infinite measure: that $\mu : C_c(X) \to \mathbb{C}$ is unbounded with respect to the supremum norm.

3. **(Uniqueness of Haar measure)** Let $G$ be a locally compact topological group and let $\mu_1, \mu_2$ be a left Haar measure on $G$.
   (a) Given $f \in C_c(G)$ show that $f$ is uniformly continuous: for any $\varepsilon > 0$ there is an open subset $U$ such that for all $x \in G$, $u \in U$ we have $|f(xu) - f(x)| < \varepsilon$. Furthermore, we can choose $U$ so that $\text{supp}(f) U$ is contained in any fixed compact set $K$. 
(b) Let $\chi \in C_c(U)$ be positive such that $\mu(\chi) = 1$ and let $(f \ast \chi)(x) = \int_G f(xu)\chi(u) \, d\mu_1(u)$. Show that $\|f \ast \chi - f\|_\infty \leq \varepsilon$ and hence
\[
\left| \int \, d\mu_2(x) \, \int \, d\mu_1(u) f(xu) \chi(u) - \int \, d\mu_2(x) \, f(x) \right| \leq \varepsilon \mu_2(K).
\]
(c) Changing variables on the LHS show that
\[
\left| \int \, d\mu_2(x) \, f(x) - E \int \, d\mu_1(x) \, f(x) \right| \leq \varepsilon \mu_2(K)
\]
with $E = \int \chi(x^{-1}) \, d\mu_2(x) > 0$.
(d) For any $f, g \in C_c(G)$ show that
\[
|\mu_1(g) \mu_2(f) - \mu_1(f) \mu_2(g)| = 0
\]
and hence that $\mu_1$ and $\mu_2$ are proportional.

4. Fix a left Haar measure $\mu$.
(a) For $f \in C_c(G)$ and $g \in G$ let $(R_g f)(x) = f(xg)$ be the left regular representation. Show that $\mu_g(f) \overset{\text{def}}{=} \mu(R_g f)$ is also a left Haar measure on $G$. It follows that there is $\delta_G(g) \in \mathbb{R}_{>0}$ such that $\mu_g(f) = \delta_G(g^{-1}) \mu(f)$ for all $f$.
RMK The $g^{-1}$ is there so that $\mu(Ag) = \delta_G(g) \mu(A)$ for every left Haar measure $\mu$, measurable $A \subset G$ and $g \in G$.
(b) Show that $\delta_G: G \rightarrow \mathbb{R}_{>0}$ is a continuous group homomorphism.
DEF The map $\delta_G: G \rightarrow \mathbb{R}_{>0}$ is called the modular character of $G$. The group $G$ is called unimodular if $\delta_G$ is the trivial character (identically 1).
(c) Show that $\mu( f(x^{-1}) \delta(x) )$ is a right Haar measure on $G$. Conclude that $G$ is unimodular if every left Haar measure is a right Haar measure.
(d) Suppose $G$ is compact. Show that $\text{Hom}_{\text{cts}}(G, \mathbb{R}_{>0}) = \{1\}$ and conclude that $G$ is unimodular.
(e) Show that every abelian group and every discrete group is unimodular.

5. (Example of Haar measure) Let $\text{GL}_n(\mathbb{R}) = \{ g \in M_n(\mathbb{R}) \mid \det g \neq 0 \}$. Let $\mu$ be the measure on $\text{GL}_n(\mathbb{R})$ with density $\frac{1}{|\text{det}(g)|^n}$ wrt Lebesgue measure – in other words:
\[
\int f(g) \, d\mu(g) = \iint f((g_{ij})_{i,j=1}^n) \frac{1}{|\text{det}(g)|^n} \, dg_{11} \cdots dg_{nn}.
\]
Show that $\mu$ is a left- and right-invariant Haar measure.

**Supplement: Tensor products of locally convex vector spaces**

Let $X, Y$ be Banach spaces and let $X \otimes Y$ be their algebraic tensor product.

6. A cross norm on $X \otimes Y$ is a norm such that
\[
\forall x \in X, y \in Y : \|x \otimes y\| = \|x\|_X \|y\|_Y
\]
\[
\forall x' \in X', y' \in Y' : \|x' \otimes y'\| = \|x'\|_{X'} \|y'\|_{Y'}
\]
(a) Show that $\|t\|_\pi = \inf \left\{ \sum_{i=1}^r \|x_i\|_X \|y_i\|_Y \mid t = \sum_{i=1}^r x_i \otimes y_i \right\}$ defines a norm on $X \otimes Y$, and that $\|t\|_\pi \geq \|t\|$ for all cross norms $\|\cdot\|$. 

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(b) Show that \( \|t\|_\varepsilon = \sup \{|(x' \otimes y')(t)| x' \in X', y' \in Y, \|x'\|_{X'} = \|y'\|_{Y'} = 1 \} \) defines a norm on \( X \otimes Y \), and that \( \|t\|_\varepsilon \leq \|t\| \) for all cross norms \( \|\cdot\| \).

(c) Let \( X \otimes_\varepsilon Y, X \otimes_\pi Y \) be the completions of \( X \otimes Y \) with respect to these norms. Obtain a continuous inclusion \( X \otimes_\varepsilon Y \hookrightarrow X \otimes_\pi Y \).

RMK In general this is not an isomorphism.