Lior Silberman’s Math 535, Problem Set 1b: Analysis

Haar measure

Let $X$ be a locally compact topological space. Write $C(X)$ for the space of continuous real-valued functions on $X$, and for $f \in C(X)$ write $\|f\|_\infty = \sup \{|f(x)| : x \in X\}$. It is well-known that the subspace $C_b(X) = \{ f \in C(X) \mid \|f\|_\infty < \infty \}$ is complete in the supremum norm and that it contains the subspace $C_c(X)$ of compactly supported functions.

**Definition.** A Radon measure on $X$ is a linear functional $\mu : C_c(X) \to \mathbb{C}$ such that $\mu(f) \geq 0$ if $f \geq 0$ (that is, if $f(x) \in \mathbb{R}_{\geq 0}$ for each $x$). If $\mu$ is a Radon measure and $f \in C_c(X)$ we often write $\int f \, d\mu$ instead of $\mu(f)$.

1. (Preliminaries)
   (a) Show that the closure of $C_c(X)$ in $C_b(X)$ is the space $C_0(X)$ of functions vanishing at infinity (continuous functions $f$ such that for all $\epsilon > 0$ there is a compact set $K \subset X$ such that $|f(x)| < \epsilon$ if $x \notin K$).
   (b) Let $X'$ be a locally compact topological space. Write $\{ F \subset X' \mid \exists U : U \supset F \}$ for an appropriately defined (left) Haar measure on $X'$.
   (b) In particular, suppose $Y$ is compact. Show that a Radon measure on $Y$ is a bounded linear functional on $C(Y) = C_b(Y) = C_c(Y)$.

2. Let $G$ be a locally compact topological group.
   (a) Let $f, f' \in C_c(G)$ be non-negative, and let $U \subset G$ be open. Set
       $$(f : U) = \inf \left\{ \sum_{i=1}^{n} \alpha_i \mid \alpha_i \geq 0, f \leq \sum_{i=1}^{n} \alpha_i \cdot 1_U \right\}.$$ 
       Show that $0 \leq (f : U) < \infty$. Assuming $f' \neq 0$ show that $(f : U) \leq (f' : U) (f : f')$ for an appropriately defined $(f : f')$ which is independent of $U$.
   (b) Let $\mathcal{N}$ be the set of open neighbourhoods of the identity in $G$; for $U \in \mathcal{N}$ set $F_U = \{ V \in \mathcal{N} \mid V \subset U \}$. Show that $\mathcal{F} = \{ S \subset \mathcal{N} \mid \exists U : S \supset F_U \}$ is a filter on $\mathcal{N}$ (that is, if $S_1, S_2 \in \mathcal{F}$ and $T \subset \mathcal{N}$ then $S_1 \cap S_2, S_1 \cup T \in \mathcal{F}$). Show that for any $V \in \mathcal{N}$ there is $S \in \mathcal{F}$ with $V \notin S$ (“$\mathcal{F}$ is not contained in any principal filter”). Let $\omega \in \mathcal{N}$ be a maximal filter containing $cF$.
   (c) Fix $f_0 \in C_c(G)$ which is non-negative and non-zero. Show that $\mu(f) \overset{\text{def}}{=} \lim_{U \in \mathcal{N}} \omega(f : U)$ extends to a $G$-invariant Radon measure on $G$. Such $\mu$ is called a (left) Haar measure on $G$.
   (d) Suppose $G$ is non-compact. Show that $\mu$ is an infinite measure: that $\mu : C_c(X) \to \mathbb{C}$ is unbounded with respect to the supremum norm.

3. Let $G$ be a locally compact topological group and let $\mu_1, \mu_2$ be two (left) Haar measures on $G$.

4. (Example of Haar measure) Let $\text{GL}_n(\mathbb{R}) = \{ g \in M_n(\mathbb{R}) \mid \det g \neq 0 \}$. Let $\mu$ be the measure on $\text{GL}_n(\mathbb{R})$ with density $\frac{1}{|\det(g)|^n}$ wrt Lebesgue measure – in other words:
   $$\int f(g) \, d\mu(g) = \int \left( \prod_{i=1}^{n} f(g_{i,j}) \right) \frac{1}{|\det(g)|^n} \prod_{i=1}^{n} dg_{i1} \cdots dg_{nn}.$$ 
   Show that $\mu$ is a left- and right-invariant Haar measure.
Supplement: Tensor products of locally convex vector spaces

Let $X, Y$ be Banach spaces and let $X \otimes Y$ be their algebraic tensor product.

5. A cross norm on $X \otimes Y$ is a norm such that
\[
\forall x \in X, y \in Y : \|x \otimes y\| = \|x\|_X \|y\|_Y \\
\forall x' \in X', y' \in Y' : \|x' \otimes y'\| = \|x'\|_{X'} \|y'\|_{Y'}
\]
(a) Show that $\|t\|_\pi = \inf \{ \sum_{i=1}^r \|x_i\|_X \|y_i\|_Y \mid t = \sum_{i=1}^r x_i \otimes y_i \}$ defines a norm on $X \otimes Y$, and that $\|t\|_\pi \geq \|t\|$ for all cross norms $\|\cdot\|$.
(b) Show that $\|t\|_\varepsilon = \sup \{ |(x' \otimes y')(t)| \mid x' \in X', y' \in Y, \|x'\|_{X'} = \|y'\|_{Y'} = 1 \}$ defines a norm on $X \otimes Y$, and that $\|t\|_\varepsilon \leq \|t\|$ for all cross norms $\|\cdot\|$.
(c) Let $X \otimes \varepsilon Y, X \otimes \pi Y$ be the completions of $X \otimes Y$ with respect to these norms. Obtain a continuous inclusion $X \otimes \varepsilon Y \hookrightarrow X \otimes \pi Y$.

RMK In general this is not an isomorphism.