These are rough notes for the Spring 2018 course. Solutions to problem sets were posted on an internal website.
Contents

Introduction 4
  0.1. Administrivia 4

Chapter 1. Basics: Locally compact groups and their representations 5
  1.1. Topological groups 5
  1.2. Representation Theory 5
  1.3. Compact groups: the Peter–Weyl Theorem 7
  1.4. Locally compact abelian groups 10
  1.5. $\text{SL}_2(\mathbb{R})$ 10

Chapter 2. Lie Groups and Lie Algebras 11
  2.1. Smooth manifolds 11
  2.2. Lie groups 12
  2.3. Lie Algebras and the exponential map 13
  2.4. Closed Subgroups 13

Chapter 3. Compact Lie groups 14
  3.1. The exponential map 14
  3.2. Maximal Tori 14

Chapter 4. Semisimple Lie groups 16

Chapter 5. Representation theory of real groups 17

Appendix A. Functional Analysis 18
  A.1. Topological vector spaces 18
  A.2. Quasicomplete locally convex TVS 19
  A.3. Integration 20
  A.4. Spectral theory and compact operators 20
  A.5. Trace-class operators and the simple trace formula 20

Appendix. Bibliography 21
Introduction

Lior Silberman, lior@Math.UBC.CA, http://www.math.ubc.ca/~lior
Office: Math Building 229B
Phone: 604-827-3031

0.1. Administrivia

• Problem sets will be posted on the course website.
  – To the extent I have time, solutions may be posted on Connect.
• Textbooks
  – Warner, Lee
  – Bröcker–tom Dieck, Representations of Compact Lie Groups, GTM-98
  – Knapp, Lie groups beyond an introduction
  – Knapp, Representation Theory of Semisimple Groups
• No exams.
CHAPTER 1

Basics: Locally compact groups and their representations

Remark 1. On foundations.

1.1. Topological groups

Definition 2. A topological group is a group object in the category of Hausdorff topological spaces. A homomorphism of topological groups is a continuous group homomorphism. An action of the topological group $G$ on the topological space $X$ is a group action $\cdot: G \times X \to X$ which is continuous for the product topology on $G \times X$.

Note that the regular action of $G$ on itself is a continuous action by homeomorphisms.

Example 3. $\mathbb{R}$, $\text{GL}_n(\mathbb{R})$, $\text{SL}_n(\mathbb{Q})$, $\mathbb{Q}_p$, $\mathbb{C}^X$ (X arbitrary!), etc.

Lemma 4. Suffices to assume $T_1$, that is that $\{e\} \subset G$ is closed.

Proof. By the invariance of the topology if $\{e\}$ is closed so is every point, and it is enough to separate $e$ from $g$ for every $g \neq e$. Since the group is $T_1$, the set $G \setminus \{g\}$ is open. By continuity of the map $(x,y) \mapsto xy^{-1}$ at the identity there is a neighbourhood $(e,e) \in U \times V \subset G \times G$ such that $xy^{-1} \neq g$ for all $(x,y) \in U \times V$. Then $W = U \cap V$ works.

Lemma 5. Let $H \subset G$ be a subgroup. Then the quotient topology on $G/H$ is Hausdorff iff $H$ is closed.

Proof. Let $q: G \to G/H$ be the quotient map. If $G/H$ is Hausdorff it is $T_1$ so $H = q^{-1}(H)$ is closed. Conversely, if $H$ is closed by invariance it is enough to separate $H,gH \subset G/H$. For that let $W \subset G$ be a neighbourhood of the identity such that $W^{-1}W \cap gH = \emptyset$. It then follows that $W^{-1}WH \cap gH = \emptyset$ as well. It follows that the open sets $WH$ and $WgH$ are disjoint, and hence that their (open) images in $G/H$ are disjoint.

1.2. Representation Theory

1.2.1. Continuous representations.

Definition 6. A representation $\pi$ of the topological group $G$ on the TVS $V_\pi$ is a continuous action by linear maps. A unitary representation is a representation on a Hilbert space $V_\pi$ by unitary maps.

Definition 7. Let $(\pi,V)$ and $(\sigma,W)$ be representations of $G$. An intertwining operator (or $G$-homomorphism) between them is a continuous map $f: V \to W$ such that

$$\forall g \in G : \sigma(g) \circ f = f \circ \pi(g).$$

We will write $\operatorname{Hom}_G(V,W)$ for the set of $G$-homomorphisms, $\operatorname{Rep}(G)$ for the category of representations and $G$-homomorphisms.
Lemma 8. Let \((\pi, V) \in \text{Rep}(G)\). If \(W \subset V\) is \(G\)-invariant then so is its closure \(\bar{W}\).

Definition 9. Call \((\pi, V)\) (topologically) irreducible if its only closed \(G\)-invariant subspaces are the obvious ones.

Example 10. Fix a group \(G\).

1. The trivial representation is the unique representation with \(V = \{0\}\).
2. For any reasonable function space, including \(C(G), L^2(G)\) (if \(G\) is locally compact and unimodular)

1.2.2. Constructions.

Lemma-Definition 11. Let \((\pi, V)\) and \((\sigma, W)\) be representations of \(G\).

1. For \(g \in G\) set \(\hat{\pi}(g) = \pi(g)^{-1}\). Then \(\hat{\pi}\) defines a representation of \(G\) on the continuous dual \(V'\).
2. Endowing \(V \oplus W\) with the product topology, setting \((\pi \oplus \sigma)(g) = \pi(g) \oplus \sigma(g)\) defines a representation.
3. Suppose \(U \subset V\) is a \(G\)-invariant closed subspace. Then setting \(\bar{\pi}(g)(v + U) = \pi(g)v + U\) defines a continuous representation of \(G\) on \(V / U\).

Proof. Exercise.

Lemma-Definition 12 (Naive tensor product). Let \((\pi, V), (\sigma, W)\) be representations of \(G, H\) respectively. Then \(G \times H\) acts on the algebraic tensor product \(V \otimes W\) by \((\pi \otimes \sigma)(g, h) \overset{\text{def}}{=} \pi(g) \otimes \sigma(h)\).

Remark 13. When \(V, W\) are finite-dimensional so is \(V \otimes W\) and there is no problem with the topology.

1.2.3. Matrix coefficients.

Definition 14. Let \((\pi, V)\) be a representation of \(G\). A matrix coefficient of \(V\) is any function \(\Phi_{v, v'}(g) = \langle \pi(g)v, v' \rangle\) where \(v \in V, v' \in V'\).

Remark 15. It is always the case that \(\Phi_{v, v'} \in C(G)\). Further analytic properties of the matrix coefficients (smoothness and decay) are very important.

Lemma 16. The map \((v, v') \mapsto \Phi_{v, v'}\) is bilinear; the resulting map \(V \otimes \bar{V} \to C(G)\) is an intertwining operator where \(G \times G\) acts on \(C(G)\) the right by \((g_1, g_2) \cdot f)(x) = f(g_2^{-1}xg_1)\).

Proof. We only prove the last claim:

\[
\Phi_{\pi(g_1)v, \pi(g_2)v'}(x) = \langle \pi(x)\pi(g_1)v, \pi(g_2^{-1})v' \rangle = \langle \pi(g_2^{-1})\pi(x)\pi(g_1)v, v' \rangle = \langle \pi(g_2^{-1}xg_1)v, v' \rangle = \Phi_{v, v'}(g_2^{-1}xg_1).
\]
Remark 17. We see that abstract representations have concrete models.

Definition 18. Call an irrep \((\pi, V)\) discrete series if it is isomorphic to an irreducible subrepresentation of the regular representation of \(G\).

Example 19. Suppose \((\pi, V)\) is unitarizable, in that there is a \(G\)-invariant continuous Hermitian product on \(V\) (so that the completion is a Hilbert space). Equipping \(V'\) with the dual inner product, which is also invariant, we see that the matrix coefficients of \(\pi\) are bounded.

1.3. Compact groups: the Peter–Weyl Theorem

In this section \(G\) is a compact group, equipped with its probability Haar measure \(dg\).

1.3.1. Finite-dimensional representations: Schur orthogonality. Fix a representation \((\pi, V)\) of \(G\) where \(V\) is finite-dimensional.

Lemma 20 (Unitarity). There is a \(G\)-invariant Hermitian product on \(V\).

Proof. Let \((\cdot, \cdot)\) be any Hermitian product on \(V\), and for \(u, v \in V\) set
\[
\langle u, v \rangle = \int_G (\pi(g)u, \pi(g)v) \, dg
\]
where \(dg\) is the probability Haar measure on \(G\).

Corollary 21. Let \(W \subset V\) be an invariant subspace. Then it has a complement: another invariant subspace \(W^\perp\) such that \(V = W \oplus W^\perp\).

Proof. Take the orthogonal complement wrt an invariant Hermitian product.

The following should be compared with the spectral theorem.

Theorem 22. Every finite-dimensional representation is a direct sum of irreducible subspaces.

Proof. Let \(U \subset V\) be maximal wrt inclusion among all subspaces which are direct sums of irreducibles. If \(U \neq V\) then \(U^\perp\) is non-trivial; let \(W \subset U^\perp\) be a non-zero invariant subspace of minimal dimension. Then \(W\) is necessarily irreducible and \(U \oplus W\) is the direct sum of irreducibles, a contradiction.

Problem 23. Isomorphism as abstract, or as unitary, representations?

Proposition 24 (Schur’s Lemma). Let \((\pi, V), (\sigma, W)\) be finite-dimensional irreducible representations of \(G\). Then
\[
\Hom_G(V, W) \simeq \begin{cases} 
\mathbb{C} & \pi \simeq \sigma \\
0 & \pi \not\simeq \sigma
\end{cases}
\]

Proof. Since the kernel and image of an intertwining operator are invariant subspaces, any non-zero \(G\)-homomorphism from an irrep is injective and to an irrep is surjective. In particular, if \(\pi, \sigma\) are non-isomorphic they support no non-zero maps between them. It remains to compute \(\Hom_G(V, V)\). For this let \(T \in \Hom_G(V, V)\), so that \(\pi(g)T = T \pi(g)\) for all \(g \in G\). Since \(\mathbb{C}\) is algebraically closed, \(T\) has at least one eigenvalue \(\lambda\); let \(V_\lambda = \text{Ker}(T - \lambda)\), a non-trivial subspace of \(V\). Then for any \(v \in V_\lambda\) we have \((T - \lambda)(\pi(g)v) = \pi(g)((T - \lambda)v) = 0\) so that \(\pi(g)v \in V_\lambda\) as well. It follows that \(V_\lambda \subset V\) is a \(G\)-invariant subspace, and hence that \(V_\lambda = V\) and \(T = \lambda \text{Id}\).
Now let \((\pi, V)\) be finite-dimensional. Every matrix coefficient of \(\pi\) is a continuous function on the compact space \(G\), hence square-integrable.

**Proposition 25** (Schur Orthogonality). Let \(\pi, \sigma \in \text{Rep}(G)\) be finite-dimensional irreps.

1. Any two matrix coefficients of \(\pi, \sigma\) are orthogonal if \(\pi, \sigma\) are non-isomorphic.
2. Let \(d_\pi = \dim V_\pi\). Then for any \(v, v' \in V\) and \(w, w' \in V'\) we have

\[
\left\langle \Phi^{\pi}_{u, u'}, \Phi^{\pi}_{v, v'} \right\rangle_{L^2(G)} = \frac{1}{d_\pi} \left\langle v, u' \right\rangle \left\langle u, v' \right\rangle
\]

**Proof.** Let \(T: V \to W\) be any linear map, and let

\[
\tilde{T} = \int_G \sigma(g^{-1}) T \pi(g) \, dg.
\]

Then

\[
\tilde{T} \pi(h) = \int_G \sigma(g^{-1}) T \pi(gh) \, dg = \int_G \sigma(hg^{-1}) T \pi(g) \, dg = \sigma(h) \tilde{T}.
\]

It follows that \(\tilde{T} \in \text{Hom}_G(V, W)\). Next, for any \(v \in V, v' \in V', w \in W, w' \in W'\) let \(T = |v\rangle \langle v'|\). Then

\[
\left\langle w' \mid \tilde{T} \mid v \right\rangle = \int_G \langle w' \mid \sigma(g^{-1}) |w \rangle \langle v' \mid \pi(g) |v \rangle \, dg
\]

\[
= \int_G d_g \langle w | \sigma(g) | w' \rangle \langle v' | \pi(g) | v \rangle = \left\langle \Phi^\sigma_{w, w'} \Phi^\pi_{v, v'} \right\rangle_{L^2(G)},
\]

where we have identified \(W'\) with \(W\) via the Riesz representation theorem and the inner product.

1. Suppose \(\pi, \sigma\) are non-isomorphic. Then \(\tilde{T} = 0\) and the two matrix coefficients are orthogonal.

2. Suppose \(V = W, \pi = \sigma\). Then \(\tilde{T} = \lambda \text{Id}\) for some \(\lambda \in \mathbb{C}\). Normalizing the Haar measure on \(G\) to be a probability measure, we see that \(\tilde{T}\) is the average of conjugates of \(T\) so

\[
d_\pi \lambda = \text{Tr} \tilde{T} = \text{Tr} T = \left\langle v', w \right\rangle.
\]

Solving for \(\lambda\) it follows that

\[
\left\langle \Phi^\pi_{w', w}, \Phi^\pi_{v, v'} \right\rangle_{L^2(G)} = \left\langle w' \mid \tilde{T} \mid v \right\rangle = \lambda \left\langle w' \mid \text{Id} \mid v \right\rangle
\]

\[
= \frac{1}{d_\pi} \left\langle w', v \right\rangle \left\langle v', u \right\rangle.
\]

\(\Box\)

**Corollary 26.** \(\langle \chi_{\pi}, \chi_{\sigma} \rangle_{L^2(G)} = \delta_{\pi \simeq \sigma}\).

**Corollary 27.** For each finite-dimensional irrep \(\pi\) let \(C(\pi)\) be the space of matrix coefficients of \(\pi\). Then

\[
\bigoplus_{\pi} C(\pi) \subset L^2(G)
\]
is an orthogonal direct sum.

1.3.2. Infinite-dimensional representations and the Peter–Weyl Theorem. Let \((\pi, V)\) be a continuous representation of the locally compact group \(G\) on the quasi-complete locally convex TVS \(V\).

**Definition 28.** For \(f \in C_c(G)\) and \(v \in V\) set \(\pi(f)v\) by

\[
\pi(f)v = \int_G f(g)\pi(g)v\,dg.
\]

**Lemma 29.** \(\pi(f) : V \to V\) is a continuous linear map, and \(f \mapsto \pi(f)\) is a continuous algebra homomorphism \(C_c(G) \to \text{End}(V)\) where \(C_c(G)\) is equipped with the convolution product and the direct limit topology.

**Proof.** Scaling, we may assume \(|f(g)| \leq 1\) for all \(g\). Let \(U \subset V\) be a closed convex neighbourhood of zero. Then for each \(g \in \text{supp}(f)\) there are neighbourhoods \(g \in W_g \subset G\) and (convex) \(0 \in U_g \subset V\) such that \(\pi(x)u \in \frac{1}{\text{vol}(\text{supp}(f))} U\) for all \(x \in W_g, u \in U_g\). Covering \(\text{supp}(f)\) with \(\bigcup_{i=1}^r W_{g_i}\) and setting \(\bar{U} = \cap_{i=1}^r U_{g_i}\), we see that for all \(g \in \text{supp}(f)\) and \(v \in \bar{U}\), \(f(g)\pi(g)v \in \frac{1}{\text{vol}(\text{supp}(f))} U\). It follows that \(\pi(f)v \in U\).

Rest proved similarly. \(\square\)

**Corollary 30.** Let \(\{f_n\} \subset C_c(G)\) be an approximate identity. Then \(\pi(f_n)v \to v\).

**Example 31 (Smoothing).** Let \(V \subset L^2(G)\) be a closed \(G\)-invariant subspace. Then \(V \cap C(G)\) is dense in \(G\).

**Proof.** It suffices to show that \(\pi(f)\varphi \in C(G)\) for each \(f \in C_c(G), \varphi \in L^2(G)\). Indeed,

\[
(\pi(f)\varphi)(x) = \int f(g)\varphi(g^{-1}x)\,dg
\]

so that

\[
|(\pi(f)\varphi)(x) - (\pi(f)\varphi)(y)| = \left|\int \delta(g)\left(f(xg^{-1}) - f(yg^{-1})\right)\varphi(g)\,dg\right|
\]

\[
\leq \left\|\delta(g)\left(f(xg^{-1}) - f(yg^{-1})\right)\right\|_{L^2(G)}\|\varphi\|_{L^2(G)}
\]

\[
\xrightarrow{y \to x} 0
\]

since \(f\) is uniformly continuous and \(\delta\) is bounded on any compact set.

Suppose now that \(G\) is compact. \(\square\)

**Theorem 32 (Peter–Weyl I).** We have

\[
L^2(G) = \bigoplus_{\pi} \hat{C}(\pi).
\]

**Proof.** Let \(V = (\bigoplus_{\pi} \hat{C}(\pi))^\perp\) and suppose \(V \neq \{0\}\). Then there is a non-zero \(f \in V \cap C(G)\). Translating and rescaling we may assume \(f(e) = 1\). Averaging we may assume \(f\) is conjugation-invariant, and replaing \(f\) by \(f(g) + f(g^{-1})\) (so that \(f(e) = 2\)) we may assume \(f(g) = f(g^{-1})\).
Now let \( f^\dagger \) be the inverse of matrix coefficients. Then \( f^\dagger \in V \) and \( f = f^\dagger \circ f \) is non-zero since \( F(e) = \| f \|^2 \). Then convolution with \( F \) is a self-adjoint compact operator on the Hilbert space \( V \). It is non-zero since \( f \circ f^\dagger = (f \circ f^\dagger)^\dagger = (f \circ f^\dagger) \). Also, convolution with \( F \) commutes with the \( G \times G \)-action on \( L^2(G) \). Let \( V_\lambda \) be any finite-dimensional eigenspace (\( \lambda \neq 0 \)); it contains an irrep \( W \) for the \( G \)-action. Let \( \Phi_{w,w'} \) be any matrix coefficient of \( W \). Then \( F \) is orthogonal to all translates of \( \Phi \), so \( F \circ \Phi = 0 \). On the other hand, \( F \circ \Phi \) is the matrix coefficient of \( \Phi_{F \circ w,w'} \) with \( F \circ w = \lambda w \), so \( F \circ \Phi = \lambda \Phi \) — a contradiction. 

**COROLLARY 33 (Peter–Weyl II).** \( \bigoplus \pi C_c(\pi) \) is dense in \( C(G) \).

**PROOF.** Since the matrix coefficients of the tensor product are products, this is a subalgebra closed under complex conjugation and it suffices to show it separates the points. By \( G \)-invariance it suffices to separate points from the identity.

For this consider \( \bigcap \pi \mathrm{Ker}(\pi) \). Every \( f \in L^2(G) \) is invariant by this closed subgroup, so it’s trivial. It follows that for any \( g \in G \) there is \( \pi \) such that \( \pi(g) \neq \mathrm{id} \). Let \( v \in V_\pi \) be of norm 1 such that \( \pi(g)v \neq v \). Then by unitarity \( \langle \pi(g)v, v \rangle \neq 1 \) and hence \( \Phi_{v,v}(g) \neq 1 \).

**THEOREM 34 (Peter–Weyl II).** Every irrep of \( G \) is finite-dimensional; for any representation \( V_K \) is dense in \( V \).

1.4. Locally compact abelian groups

1.5. \( SL_2(\mathbb{C}) \)
CHAPTER 2

Lie Groups and Lie Algebras

2.1. Smooth manifolds

2.1.1. Manifolds.

Definition 35. Let \( U \subset \mathbb{R}^n \) be open. Then \( C^\infty(U; \mathbb{R}^m) \) is the set of infinitely differentiable \( \mathbb{R}^m \)-valued functions on \( U \).

Definition 36. A smooth \( n \)-manifold is a topological space ...

Example 37. \( \mathbb{R}^n \),

Lemma 38. If \( m \neq n \), \( \mathbb{R}^m, \mathbb{R}^n \) are not locally homeomorphic so for a connected manifold the dimension need not be assumed constant.

Definition 39. Let \( M, N \) be smooth manifolds. A map \( f : M \to N \) is smooth if ...

2.1.2. Tangent and cotangent spaces.

Definition 40. A Lie algebra over \( k \) is a \( k \)-vector space \( g \) together with a bilinear form \([·,·] : g \times g \to g\) satisfying:

1. (alternating) \([X,X] = 0\)
2. (Jacobi identity) \([[X,Y],[Z]] + [[Y,Z],[X]] + [[Z,X],[Y]] = 0\).

Example 41 (Standard constructions). Let \( A \) be an associative \( k \)-algebra. We get two natural Lie algebras from it:

1. \( A \) itself, equipped with \([a,b] = ab - ba\).
2. Call \( d \in \text{End}_k(\mathbb{A}) \) a derivation if \( d(ab) = d(a)b + ad(b) \). Then the space \( \mathcal{D}_A \) of derivations is a Lie subalgebra of \( \text{End}_k(\mathbb{A}) \).
3. One canonical example: \( A = C^\infty(M) \); then \( \mathcal{D}_A \) is called the set of vector fields on \( M \).

Proposition 42 (Canonical sheaf).

1. (Localization) Let \( X \) be a vector field on \( M \) and let \( f_1, f_2 \in C^\infty(M) \) agree on a neighbourhood of \( p \in M \). Then \( X f_1(p) = X f_2(p) \).
2. Let \( U \subset M \) be open. Then for a vector field \( X \) on \( M \), \( f \in C^\infty(U) \) and \( p \in U \) let \( h \in C^\infty(U) \) such that \( h \equiv 1 \) near \( p \) and set \( (X \upharpoonright_U f)(p) = (X(hf))(p) \) (note that \( hf \in C^\infty(M) \)). Then \( X \upharpoonright_U \) is a vector field on \( U \) and \( X \mapsto X \upharpoonright_U \) is a map of lie algebras.
3. (Patching) Let \( \{U_i\}_{i=1}^r \) be an open cover of \( M \). Let \( X, Y \) be a vector fields on \( M \) and suppose that \( X \upharpoonright_{U_i} = Y \upharpoonright_{U_i} \) for all \( i \) then \( X = Y \).
4. (Gluing) Let \( \{U_i\}_{i=1}^r \) be an open cover of \( M \) and suppose given for each \( i \) a vector field \( X_i \) on \( U_i \) such that \( X_i \upharpoonright_{U_i \cap U_j} = X_j \upharpoonright_{U_i \cap U_j} \) for all \( i, j \). Then there is a vector field \( X \) on \( M \) such that \( X_i = X \upharpoonright_{U_i} \).

Lemma-Definition 43. \( I_p = \{ f \in C^\infty(M) \mid f(p) = 0 \} \) is a maximal ideal of \( C^\infty(M) \).
LEMMA-DEFINITION 44. The contangent space $T_p^*M = I_p/I^2_p$ is a vector space of dimension $n$ and $\bigcup_{p \in M} T_p^*M$ is a vector bundle.

LEMMA-DEFINITION 45 (The tangent space). The linear dual $T_pM = \text{Hom}_\mathbb{R}(T_p^*M, \mathbb{R})$ is called the tangent space. The resulting bundle is called the tangent bundle.

1. The pairing $(X, f) \mapsto Xf(p)$ associates to each vector field $X$ a linear functional on $T_p^*X$.
2. The map vector fields $\rightarrow (T_p^*)'$ is surjective.

EXERCISE 46. $T_pM$ is also the space of derivations on the algebra of germs of smooth functions at $p$.

2.1.3. Derivatives of maps.

LEMMA-DEFINITION 47. $df$

DEFINITION 48. A smooth map $f : M \rightarrow N$ is a:
1. Submersion if $df_p$ is injective for every $p \in M$.
2. Local embedding if it is a submersion and for every $p \in U \subset M$ there is $f(p) \in V \subset N$ such that $f \mid_{f^{-1}(V)}$ is a homeomorphism onto its image (with the relative topology).
3. An embedding if it is an injective immersion which is a homeomorphism onto its image.
4. A diffeomorphism if it has a smooth inverse.

DEFINITION 49. A parametrized submanifold of $N$ is a pair $(M, f)$ where $f : M \rightarrow N$ is an injective submersion. Two parametrizations $(M_1, f_1), (M_2, f_2)$ are equivalent if they are conjugate by a diffeomorphism of $M_1, M_2$. A submanifold of $N$ is an equivalence class.

2.1.4. The exponential map.

2.2. Lie groups

DEFINITION 50. A Lie group is a group object in the category of smooth manifolds, in other words a smooth manifold $G$ together with smooth maps $\cdot : G \times G \rightarrow G$ and $^{-1} : G \rightarrow G$ such that $(G, \cdot, ^{-1})$ is an abstract group. A homomorphism of Lie group is an abstract homomorphism which is also a smooth map.

DEFINITION 51. An action of a Lie group $G$ on a smooth manifold $M$ is a smooth map $\cdot : G \times M \rightarrow M$ which is a group action.

DEFINITION 52. A Lie subgroup $H$ of the Lie group $G$ is a subgroup $H < G$ which is also a submanifold, in other words the image of an injective immersion of Lie groups.

EXAMPLE 53. Lie of irrational slope on a torus.

REMARK 54. There is some play in the joints here.
1. Enough to assume $C^2$, and may assume real-analytic (any $C^2$ structure is compatible with a unique smooth, even real-analytic, structure).
2. Sophus Lie actually considered local Lie group actions.
2.3. Lie Algebras and the exponential map

Fix a vector space $k$.

**Definition 55.** A Lie algebra over $k$ is a $k$-vector space $\mathfrak{g}$ together with a bilinear form $\left[ \cdot , \cdot \right] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

1. (alternating) $\left[ X, X \right] = 0$
2. (Jacobi identity) $\left[ \left[ X, Y \right], Z \right] + \left[ \left[ Y, Z \right], X \right] + \left[ \left[ Z, X \right], Y \right] = 0$.

**Example 56 (Standard constructions).** Let $A$ be an associative $k$-algebra. We get two natural Lie algebras from it:

1. $A$ itself, equipped with $\left[ a, b \right] = ab - ba$.
2. Call $d \in \text{End}_{k,\text{vsp}}(A)$ a derivation if $d(ab) = d(a)b + ad(b)$. Then the space $\mathcal{D}_A$ of derivations is a Lie subalgebra of $\text{End}_{k,\text{vsp}}(A)$.
3. One canonical example: $A = C^\infty(M)$; then $\mathcal{D}_A$ is the set of vector fields on $M$.

2.4. Closed Subgroups

**Theorem 57 (Cartan 1930).** Let $H < G$ be a closed subgroup. Then $H$ is a Lie subgroup (in particular, a submanifold of $G$).

**Proof.** Let $\mathfrak{h} = \{ X \in \mathfrak{g} | \forall t \in \mathbb{R} : \exp(tX) \in H \}$. Then $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra. □

**Lemma 58.** Let $V \subset G$ be a small enough neighbourhood of the identity. Then

**Theorem 59.** Let $H$ be a closed subgroup of $G$. Then $G/H$ has a unique orbifold structure such that $\pi : G \to G/H$ is a submersion. Furthermore, the regular action of $G$ on $G/H$ is a Lie group action.
CHAPTER 3

Compact Lie groups

3.1. The exponential map

For more on Riemannian metrics see PS ???

**Lemma 60.** A connected compact Lie group has a bi-invariant Riemannian metric

**Remark 61.** The map $g \mapsto g^{-1}$ is an isometry of this metric. In other words, we have a symmetric space. (c.f. PS ??)

**Proposition 62.** Fix a bi-invariant metric on $G$. Then the Riemannian and Lie exponential maps agree.

**Proof.** Let $\gamma(t)$ be a Riemannian geodesic based at the origin. Then $t \mapsto \gamma(t_0 + t)$, $t \mapsto \gamma(t_0)\gamma(t)$ and $t \mapsto \gamma(t)\gamma(t_0)$ are also geodesics (because the group acts by isometries) which meet at $t = 0$ and have the same derivative at that time. It follows that $\gamma(t_0 + t) = \gamma(t_0)\gamma(t)$, that is that the geodesic is a one-parameter subgroup. 

**Corollary 63.** The exponential map of a connected compact Lie group is surjective.

**Corollary 64.** The intersection of two connected subgroups is connected.

**Proof.** The Lie algebra of the intersection is the intersection of the Lie algebras.

3.2. Maximal Tori

3.2.1. Tori.

**Definition 65.** A torus is a Lie group isomorphic to $(\mathbb{R} / \mathbb{Z})^k$.

- Aut$(T)$
- Characters and cocharacters
- Homomorphic image of a torus is a torus.

Every finite-dimensional connected compact abelian group is a torus.

Let $T \subset G$ be a torus.

**Lemma 66.** The centralizer of $T$ is connected, and equals the connected component of its normalizer.

**Proof.** Let $t \in T$ have dense orbit, so that $Z_G(t) = Z_G(T)$. Embed $G$ in $U(n)$. Wlog the image of $t$ is diagonal and then $Z_{U(n)}(t)$ is block-diagonal, in particular connected. It follows that $Z_G(T) = Z_G(t) = G \cap Z_{U(n)}(t)$ is connected.

Next, let $N_G(T)$ act on $T$ by conjugation. This gives a continuous homomorphism $N_G(T) \to \text{Aut}(T) \cong \text{GL}_r(\mathbb{Z})$. Since the latter group is discrete, the connected component is in the kernel and hence $N_G(T)^o \subset Z_G(T)$. Since $Z_G(T) \subset N_G(T)$ is connected we also have the reverse inclusion.
3.2.2. Maximal tori. Fix a connected compact Lie group $G$.

**Definition 67.** A *maximal torus* in $G$ is a torus in $G$, maximal wrt inclusion.

**Lemma 68.** *Every element* $g \in G$ *is contained in a torus.*

**Proof.** Suppose $g = \exp(x)$ for $x \in \mathfrak{g}$. Then the closure of $\{\exp(tx)\}_{t \in \mathbb{R}}$ is an abelian subgroup, hence a torus. □

**Corollary 69.** *Every element of* $g$ *is contained in a maximal torus.*

Fix a maximal torus $T$.

**Corollary 70.** $N_G(T)^{\circ} = Z_G(T) = T$.

**Proof.** Let $g \in Z_G(T)$ not belong to $T$. Then there is a torus $S \subset Z_G(T)$ such that $g \in S$. Then $ST$ is a torus properly containing $T$. □

**Definition 71.** The *Weyl group* of $G$ is $W(G : T) \overset{\text{def}}{=} N_G(T)/Z_G(T) = N_G(T)/T$.

**Theorem 72.** *All maximal tori of* $G$ *are conjugate.*

**Proof.** Let □
CHAPTER 4

Semisimple Lie groups
CHAPTER 5

Representation theory of real groups
APPENDIX A

Functional Analysis

In this appendix we review the basics of topological vector spaces. References include TVS.

A.1. Topological vector spaces

Let $K$ be a non-discrete complete valued field

**Definition 73.** A topological vector space is a vector space $V$ over $K$ equipped with a topology so that $(V, +)$ is a topological group and such that scalar multiplication is a continuous map $\cdot : K \times V \to V$.

**Proposition 74.** A finite-dimensional $K$-vector space has a unique topology making it into a TVS. In particular, if $V$, $W$ are TVS with $V$ finite-dimensional then $\text{Hom}_K(V, W) = \text{Hom}_{cts}(V, W)$ and if $V \subset W$ then $V$ is closed and complete. If $K$ is locally compact then a TVS over $K$ is locally compact iff it is finite-dimensional.

**Definition 75.** Fix a TVS $V$. Call $C \subset V$:

1. Balanced, if $\alpha v \in C$ for all $x \in C$, $|\alpha| \leq 1$
2. Absorbing, if $\bigcup_{t > 0} tC = V$ (that is, for all $v \in V$ there are $u \in C$ and $t > 0$ such that $tu = v$.
3. Bounded, if for every open neighbourhood $W \ni 0$ there is $t > 0$ such that $C \subset tW$.
4. Totally bounded, if for every open neighbourhood $W \ni 0$ there is a finite set $\{u_i\}_{i=1}^n \subset V$ such that $C \subset \bigcup_i (u_i + W)$.

**Lemma 76.** Every finite subset of a TVS is bounded.

**Lemma 77.** Every TVS has a basis neighbourhoods of 0 which are balanced.

**Definition 78.** A net $\{x_\alpha\}_{\alpha \in D} \subset V$ is called a Cauchy net if for every neighbourhood $W \ni 0$ there is $\delta \in D$ such that if $\alpha, \beta \geq \delta$ then $x_\alpha - x_\beta \in W$. $X \subset V$ is complete if every Cauchy net in $X$ converges to a limit in $X$. $V$ is quasi-complete if every closed bounded subset of $X$ is complete.

**Lemma 79.** In a quasi-complete TVS every totally bounded subset is relatively compact.

**Assumption 80.** $K = \mathbb{R}$ or $\mathbb{C}$.

**Definition 81.** Fix a TVS $V$. Call $C \subset V$ convex, if $tu + (1 - t)v \in C$ for all $u, v \in C$, $t \in [0, 1]$. Call $V$ locally convex if any neighbourhood of 0 contains a convex neighbourhood of zero.

**Proposition 82.** A TVS is locally convex iff its topology is determined by a family of semi-norms.

**Lemma 83.** Let $V$ be locally convex, $C \subset V$ be totally bounded. Then the convex hull and balanced convex hull of $C$ are also totally bounded.
COROLLARY 84. Let V be locally convex and quasi-complete and let $C \subset V$ be compact. Then the closed convex hull of C is compact.

DEFINITION 85. The continuous dual of $V$ is $V' \overset{\text{def}}{=} \text{Hom}_{\text{cts}}(V, K)$.

THEOREM 86 (Hahn–Banach). Let $V$ be locally convex, $U \subset E$ a subspace, $f \in U'$. Then $f$ has a continuous linear extension to $V$. In particular, $V'$ separates the points of $V$.

A.2. Quasicomplete locally convex TVS

[based on Casseleman, Garrett]

PROPOSITION 87. An inverse limit of quasi-complete spaces is quasi-complete. The direct product of a family of quasi-complete space is quasi-complete. The weak-* dual of a Banach space is quasi-complete.

Let $V$ be a locally convex TVS.

DEFINITION 88. Let $\Omega$ be a measureable space.

(1) Call $f: \Omega \to V$ weakly measurable if $\varphi \circ f: \Omega \to K$ is measurable for each $\varphi \in V'$. Let

(2) Let $\mu$ be a measure on $\Omega$ and let $f: \Omega \to V$ be weakly measurable. Call $v \in V$ the Gelfand–Pettis integral of $f$ (and write $v = \int f \, d\mu$) if for every $\varphi \in V'$ $\varphi \circ f$ is $\mu$-integrable and we have

$$
\varphi(v) = \int_\Omega \varphi \circ f \, d\mu.
$$

REMARK 89. Note that the integral clearly exists as an element of $V''$; the question is about existence as an element of $V$. Since $V'$ separates the points, it is also clear that the integral (if it exists) is unique.

THEOREM 90. Let $V$ be quasi-complete, let $\Omega$ be compact, $\mu$ a Radon measure, and let $f: \Omega \to V$ be continuous. Then $\int f \, d\mu$ exists.

PROOF. Wlog $\mu$ is a probability measure. In that case we also show $\int f \, d\mu$ lies in the closed convex hull of $f(\Omega)$.

LEMMA 91. If $V$ is finite-dimensional then $\int f \, d\mu$ exists and lies in the convex hull of $f(\Omega)$.

Write $C$ for the closed convex hull of $f(\Omega)$. For every finite $\mathcal{F} \subset V'$ consider the continuous linear map $\mathcal{F}: V \to K^\mathcal{F}$ given by $v \mapsto (\varphi(v))_{\varphi \in \mathcal{F}}$. It maps $C$ continuously onto the convex hull of the image of $\mathcal{F} \circ f$. Now $\int_\Omega (\mathcal{F} \circ f) \, d\mu$ exists in that convex hull, and we obtain a non-empty closed convex subset

$$
C_{\mathcal{F}} = \left\{ v \in C \mid \mathcal{F}(v) = \int_\Omega (\mathcal{F} \circ f) \, d\mu \right\}.
$$

Since $\bigcap_{i=1}^r C_{\mathcal{F}_i} = C_{\bigcup_{i} \mathcal{F}_i}$ we see that this family has the finite intersection property, and it follows that

$$
\bigcap_{\mathcal{F}} C_{\mathcal{F}}
$$

is non-empty. The (necessarily unique) point there is the desired integral. □
A.3. Integration

A.4. Spectral theory and compact operators

A.5. Trace-class operators and the simple trace formula
Bibliography