These are rough notes for the Spring 2018 course. Solutions to problem sets were posted on an internal website.
Contents

Introduction 4
  0.1. Administrivia 4

Chapter 1. Basics: Locally compact groups and their representations 5
  1.1. Topological groups 5
  1.2. Representation Theory 5
  1.3. Compact groups: the Peter–Weyl Theorem 7

Chapter 2. Lie Groups and Lie Algebras 12
  2.1. Smooth manifolds 12
  2.2. Lie groups 15
  2.3. Lie Algebras and the exponential map 16
  2.4. Closed Subgroups 17
  2.5. The adjoint representation 17

Chapter 3. Compact Lie groups 19
  3.1. Linearity 19
  3.2. Characters and cocharacters of tori 19
  3.3. The exponential map 20
  3.4. Maximal Tori 20
  3.5. Roots and weights 22

Chapter 4. Semisimple Lie groups 31

Chapter 5. Representation theory of real groups 32

Appendix A. Functional Analysis 33
  A.1. Topological vector spaces 33
  A.2. Quasicomplete locally convex TVS 34
  A.3. Integration 34
  A.4. Spectral theory and compact operators 34
  A.5. Trace-class operators and the simple trace formula 34
Introduction

Lior Silberman, lior@Math.UBC.CA, http://www.math.ubc.ca/~lior
Office: Math Building 229B
Phone: 604-827-3031

0.1. Administrivia

• Problem sets will be posted on the course website.
  – To the extent I have time, solutions may be posted on Connect.
• Textbooks
  – Warner, Lee
  – Knapp, *Lie groups beyond an introduction*
  – Knapp, *Representation Theory of Semisimple Groups*
• No exams.
CHAPTER 1

Basics: Locally compact groups and their representations

REMARK 1. On foundations.

1.1. Topological groups

DEFINITION 2. A topological group is a group object in the category of Hausdorff topological spaces. A homomorphism of topological groups is a continuous group homomorphism. An action of the topological group \( G \) on the topological space \( X \) is a group action \( \cdot: G \times X \to X \) which is continuous for the product topology on \( G \times X \).

Note that the regular action of \( G \) on itself is a continuous action by homeomorphisms.

EXAMPLE 3. \( \mathbb{R} \), \( \text{GL}_n(\mathbb{R}) \), \( \text{SL}_n(\mathbb{Q}) \), \( \mathbb{Q}_p \), \( \mathbb{C} \times X \) (\( X \) arbitrary!), etc.

LEMMA 4. Suffices to assume \( T_1 \), that is that \( \{e\} \subset G \) is closed.

PROOF. By the invariance of the topology if \( \{e\} \) is closed so is every point, and it is enough to separate \( e \) from \( g \) for every \( g \neq e \). Since the group is \( T_1 \), the set \( G \setminus \{g\} \) is open. By continuity of the map \( (x,y) \mapsto xy^{-1} \) at the identity there is a neighbourhood \( (e,e) \in U \times V \subset G \times G \) such that \( xy^{-1} \neq g \) for all \( (x,y) \in U \times V \). Then \( W = U \cap V \) works.

LEMMA 5. Let \( H \subset G \) be a subgroup. Then the quotient topology on \( G/H \) is Hausdorff iff \( H \) is closed.

PROOF. Let \( q: G \to G/H \) be the quotient map. If \( G/H \) is Hausdorff it is \( T_1 \) so \( H = q^{-1}(H) \) is closed. Conversely, if \( H \) is closed by invariance it is enough to separate \( H, gH \in G/H \). For that let \( W \subset G \) be a neighbourhood of the identity such that \( W^{-1}W \cap gH = \emptyset \). It then follows that \( W^{-1}WH \cap gH = \emptyset \) as well. It follows that the open sets \( WH \) and \( WgH \) are disjoint, and hence that their (open) images in \( G/H \) are disjoint.

1.2. Representation Theory

1.2.1. Continuous representations.

DEFINITION 6. A representation \( \pi \) of the topological group \( G \) on the TVS \( V_\pi \) is a continuous action by linear maps. A unitary representation is a representation on a Hilbert space \( V_\pi \) by unitary maps.

DEFINITION 7. Let \( (\pi,V) \) and \( (\sigma,W) \) be representations of \( G \). An intertwining operator (or \( G \)-homomorphism) between them is a continuous map \( f: V \to W \) such that

\[
\forall g \in G : \sigma(g) \circ f = f \circ \pi(g).
\]

We will write \( \text{Hom}_G(V,W) \) for the set of \( G \)-homomorphisms, \( \text{Rep}(G) \) for the category of representations and \( G \)-homomorphisms.
LEMMA 8. Let $(\pi, V) \in \text{Rep}(G)$. If $W \subset V$ is $G$-invariant then so is its closure $\bar{W}$.

DEFINITION 9. Call $(\pi, V)$ (topologically) \textit{irreducible} if its only closed $G$-invariant subspaces are the obvious ones.

EXAMPLE 10. Fix a group $G$.

(1) The \textit{trivial representation} is the unique representation with $V = \{0\}$.
(2) For any reasonable function space, including $C(G)$, $L^2(G)$ (if $G$ is locally compact and unimodular)

1.2.2. Constructions.

LEMMA-DEFINITION 11. Let $(\pi, V)$ and $(\sigma, W)$ be representations of $G$.

(1) For $g \in G$ set $\hat{\pi}(g) = \pi(g)^{-1}$. Then $\hat{\pi}$ defines a representation of $G$ on the continuous dual $V'$.

(2) Endowing $V \oplus W$ with the product topology, setting $(\pi \oplus \sigma)(g) = \pi(g) \oplus \sigma(g)$ defines a representation.

(3) Suppose $U \subset V$ is a $G$-invariant closed subspace. Then setting $\bar{\pi}(g)(v + U) = \pi(g)v + U$ defines a continuous representation of $G$ on $V / U$.

PROOF. Exercise. \qed

LEMMA-DEFINITION 12 (Naive tensor product). Let $(\pi, V)$, $(\sigma, W)$ be representations of $G, H$ respectively. Then $G \times H$ acts on the algebraic tensor product $V \otimes W$ by $(\pi \otimes \sigma)(g, h) \overset{\text{def}}{=} \pi(g) \otimes \sigma(h)$.

REMARK 13. When $V, W$ are finite-dimensional so is $V \otimes W$ and there is no problem with the topology.

1.2.3. Matrix coefficients.

DEFINITION 14. Let $(\pi, V)$ be a representation of $G$. A matrix coefficient of $V$ is any function

$$\Phi_{\psi, \phi'}(g) = \langle \pi(g)\psi, \phi' \rangle$$

where $\psi \in V$, $\phi' \in V'$.

REMARK 15. It is always the case that $\Phi_{\psi, \phi'} \in C(G)$. Further analytic properties of the matrix coefficients (smoothness and decay) are very important.

LEMMA 16. The map $(\psi, \phi') \mapsto \Phi_{\psi, \phi'}$ is bilinear; the resulting map $V \otimes \bar{V} \to C(G)$ is an intertwining operator where $G \times G$ acts on $C(G)$ the right by $((g_1, g_2) \cdot f)(x) = f(g_2^{-1}xg_1)$.

PROOF. We only prove the last claim:

$$\Phi_{\pi(g_1)\psi, \pi(g_2)\phi'}(x) = \langle \pi(x)\pi(g_1)\psi, \pi(g_2^{-1})\phi' \rangle$$
$$= \langle \pi(g_2^{-1})\pi(x)\pi(g_1)\psi, \phi' \rangle$$
$$= \langle \pi(g_2^{-1}xg_1)\psi, \phi' \rangle$$
$$= \Phi_{\psi, \phi'}(g_2^{-1}xg_1).$$

\qed
Remark 17. We see that abstract representations have *concrete models*.

Definition 18. Call an irrep \((\pi, V)\) *discrete series* if it is isomorphic to an irreducible subrepresentation of the regular representation of \(G\).

Example 19. Suppose \((\pi, V)\) is *unitarizable*, in that there is a \(G\)-invariant continuous Hermitian product on \(V\) (so that the completion is a Hilbert space). Equipping \(V'\) with the dual inner product, which is also invariant, we see that the matrix coefficients of \(\pi\) are bounded.

### 1.3. Compact groups: the Peter–Weyl Theorem

In this section \(G\) is a compact group, equipped with its probability Haar measure \(dg\).

#### 1.3.1. Finite-dimensional representations: Schur orthogonality

Fix a representation \((\pi, V)\) of \(G\) where \(V\) is finite-dimensional.

**Lemma 20 (Unitarity).** There is a \(G\)-invariant Hermitian product on \(V\).

**Proof.** Let \((\cdot, \cdot)\) be any Hermitian product on \(V\), and for \(u, v \in V\) set
\[
\langle u, v \rangle = \int_G \langle \pi(g)u, \pi(g)v \rangle \, dg
\]
where \(dg\) is the probability Haar measure on \(G\). \(\square\)

**Corollary 21.** Let \(W \subset V\) be an invariant subspace. Then it has a complement: another invariant subspace \(W^\perp\) such that \(V = W \oplus W^\perp\).

**Proof.** Take the orthogonal complement wrt an invariant Hermitian product. \(\square\)

The following should be compared with the spectral theorem.

**Theorem 22 (Maschke).** Every finite-dimensional representation is a direct sum of irreducible subspaces.

**Proof.** Let \(U \subset V\) be maximal wrt inclusion among all subspaces which are direct sums of irreducibles. If \(U \neq V\) then \(U^\perp\) is non-trivial; let \(W \subset U^\perp\) be a non-zero invariant subspace of minimal dimension. Then \(W\) is necessarily irreducible and \(U \oplus W\) is the direct sum of irreducibles, a contradiction. \(\square\)

**Problem 23.** Isomorphism as abstract, or as unitary, representations?

**Proposition 24 (Schur’s Lemma).** Let \((\pi, V), (\sigma, W)\) be finite-dimensional irreducible representations of \(G\). Then \(\text{Hom}_G(V, W) \simeq \begin{cases} \mathbb{C} & \pi \simeq \sigma \\ 0 & \pi \not\simeq \sigma \end{cases}\).

**Proof.** Since the kernel and image of an intertwining operator are invariant subspaces, any non-zero \(G\)-homomorphism from an irrep is injective and to an irrep is surjective. In particular, if \(\pi, \sigma\) are non-isomorphic they support no non-zero maps between them. It remains to compute \(\text{Hom}_G(V, V)\). For this let \(T \in \text{Hom}_G(V, V)\), so that \(\pi(g)T = T \pi(g)\) for all \(g \in G\). Since \(\mathbb{C}\) is algebraically closed, \(T\) has at least one eigenvalue \(\lambda\); let \(V_\lambda = \ker(T - \lambda)\), a non-trivial subspace of \(V\). Then for any \(v \in V_\lambda\) we have \((T - \lambda)(\pi(g)v) = \pi(g)((T - \lambda)v) = 0\) so that \(\pi(g)v \in V_\lambda\) as well. It follows that \(V_\lambda \subset V\) is a \(G\)-invariant subspace, and hence that \(V_\lambda = V\) and \(T = \lambda \text{Id}\). \(\square\)
Now let \((\pi, V)\) be finite-dimensional. Every matrix coefficient of \(\pi\) is a continuous function on the compact space \(G\), hence square-integrable.

**Proposition 25 (Schur Orthogonality).** Let \(\pi, \sigma \in \text{Rep}(G)\) be finite-dimensional irreps.

1. Any two matrix coefficients of \(\pi, \sigma\) are orthogonal if \(\pi, \sigma\) are non-isomorphic.
2. Let \(d_\pi = \dim V_\pi\). Then for any \(v, w \in V\) and \(v', w' \in V'\) we have

\[
\langle \Phi^\pi_{|w'\rangle, \Phi^\pi_{|v'\rangle} \rangle_{L^2(G)} = \frac{1}{d_\pi} \langle v, u' \rangle \langle u, v' \rangle
\]

**Proof.** Let \(T : V \to W\) be any linear map, and let

\[
\bar{T} = \int_G \sigma(g^{-1}) T \pi(g) \, dg.
\]

Then

\[
\bar{T} \pi(h) = \int_G \sigma(g^{-1}) T \pi(gh) \, dg = \int_G \sigma(hg^{-1}) T \pi(g) \, dg = \sigma(h) \bar{T}.
\]

It follows that \(\bar{T} \in \text{Hom}_G(V, W)\). Next, for any \(v \in V, v' \in V', w \in W, w' \in W'\) let \(T = |w\rangle \langle v'|\). Then

\[
\langle w' | \bar{T} | v \rangle = \int_G \langle w' | \sigma(g^{-1}) | w \rangle \langle v' | \pi(g) | v \rangle \, dg
\]

\[
= \int_G d\langle w | \sigma(g) | w' \rangle \langle v' | \pi(g) | v \rangle
\]

\[
= \langle \Phi^\sigma_{|w\rangle, \Phi^\sigma_{|v'\rangle} \rangle_{L^2(G)},
\]

where we have identified \(W'\) with \(W\) via the Riesz representation theorem and the inner product.

1. Suppose \(\pi, \sigma\) are non-isomorphic. Then \(\bar{T} = 0\) and the two matrix coefficients are orthogonal.
2. Suppose \(V = W, \pi = \sigma\). Then \(\bar{T} = \lambda \text{Id}\) for some \(\lambda \in \mathbb{C}\). Normalizing the Haar measure on \(G\) to be a probability measure, we see that \(\bar{T}\) is the average of conjugates of \(T\) so

\[
d_\pi \lambda = \text{Tr} \bar{T} = \text{Tr} T = \langle v', w \rangle.
\]

Solving for \(\lambda\) it follows that

\[
\langle \Phi^\pi_{|w'\rangle, \Phi^\pi_{|v'\rangle} \rangle_{L^2(G)} = \langle w' | \bar{T} | v \rangle = \lambda \langle w' | \text{Id} | v \rangle
\]

\[
= \frac{1}{d_\pi} \langle w', v \rangle \langle v', u \rangle.
\]

\(\square\)

**Corollary 26.** \(\langle \chi_\pi, \chi_\sigma \rangle_{L^2(G)} = \delta_{\pi \sim \sigma}\).

**Corollary 27.** For each finite-dimensional irrep \(\pi\) let \(C(\pi)\) be the space of matrix coefficients of \(\pi\). Then

\[
\bigoplus \pi \quad C(\pi) \subset L^2(G)
\]

8
is an orthogonal direct sum.

1.3.2. Infinite-dimensional representations and the Peter–Weyl Theorem. Let \((\pi, V)\) be a continuous representation of the locally compact group \(G\) on the quasi-complete locally convex TVS \(V\).

**Lemma-Definition 28.** TFAE for \(v \in V\), in which case we call it \(G\)-finite

1. \(\text{Span}_\mathbb{C} \{\pi(g)v\}_{g \in G} \subset V\) is finite-dimensional.
2. There is a finite-dimensional \(G\)-invariant subspace \(W \subset V\) with \(v \in W\).

Furthermore, the set \(V_K\) of \(K\)-finite vectors is a \(G\)-invariant algebraic subspace of \(V\).

**Proof.** Given (1), set \(W = \text{Span}_\mathbb{C} \{\pi(g)v\}_{g \in G}\) to get (2). Given (2), \(\text{Span}_\mathbb{C} \{\pi(g)v\}_{g \in G} \subset W\) for all \(G\)-invariant subspaces \(W\) containing \(v\). Finally, if \(v_1, v_2 \in V_K\), say with \(v_i \subset W_i\) with \(W_i\) \(G\)-inv’t and f.d. then \(\alpha v_1 + \pi(g)v_2 \in W_1 + W_2\) which is \(G\)-inv’t and f.d. \(\square\)

**Proposition 29.** In a compact group \(G\) we have \(\bigoplus \pi C(\pi) = L^2(G)_K\), where \(G\) acts on \(L^2(G)\) via the right-regular representation \((R_g f)(x) = f(xg)\).

**Proof.** Since each \(C(\pi)\) is finite-dimensional, their algebraic direct sum is contained in \(C(G)_K \subset L^2(G)_K\). Conversely, let \(W \subset L^2(G)\) be a right-\(G\)-invariant finite-dimensional subspace. By Maschke’s Theorem \(W\) is the direct sum of irreducible subspaces so without loss of generality it suffices to show \(W \subset \bigoplus \pi C(\pi)\) for irreducible \(W\).

Now let \(\{f_i\}_{i=1}^d \subset W\) be an orthonormal basis. Then for \(f \in W\) and \(g \in G\) we have \(R_g f \in W\) and hence

\[
R_g f = \sum_{i=1}^d a_i(g) f_i
\]

for some \(a_i(g) \in \mathbb{C}\). In fact,

\[
a_i(g) = \langle f_i, R_g f \rangle_{L^2(G)} = \Phi_{f_i,f}^W(g) \in C(W)
\]

and we conclude that for fixed \(g\)

\[
R_g f = \sum_{i=1}^d \Phi_{f_i,f}^W(g) f_i
\]

(the sum in \(W \subset L^2(G)\)). In other words, given \(g\) it holds for almost every \(x \in G\) that

\[
f(xg) = \sum_{i=1}^d \Phi_{f_i,f}^W(g)(x) f_i(x).
\]

If the identity held for all \(x\) we could set \(x = e\) and write \(f\) as a linear combination of matrix coefficients. To get around this difficulty consider both sides as functions on \(G \times G\). Now both sides are in \(L^2(G \times G)\), so by Fubini they are equal a.e. Applying Fubini in the other order it follows that for almost every \(x \in G\) we have \(f(xg) = \sum_{i=1}^d \Phi_{f_i,f}^W(g)(x) f_i(x)\) for almost every \(g \in G\), and that is the desired claim. \(\square\)

**Definition 30.** For \(f \in C_c(G)\) and \(v \in V\) set \(\pi(f)v\) by

\[
\pi(f)v = \int_G f(g) \pi(g)v \, dg.
\]
Lemma 31. $\pi(f) : V \to V$ is a continuous linear map, and $f \mapsto \pi(f)$ is a continuous algebra homomorphism $C_c(G) \to \text{End}(V)$ where $C_c(G)$ is equipped with the convolution product and the direct limit topology.

Proof. Scaling, we may assume $|f(g)| \leq 1$ for all $g$. Let $U \subset V$ be a closed convex neighbourhood of zero. Then for each $g \in \text{supp}(f)$ there are neighbourhoods $g \in W_{g} \subset G$ and (convex) $\emptyset \subset U_{g} \subset V$ such that $\pi(x)u \in U_{g}$ for all $x \in W_{g}$, $u \in U_{g}$. Covering supp$(f)$ with $\cup_{i=1}^{n}W_{g_{i}}$ and setting $\bar{U} = \cap_{i=1}^{n}U_{g_{i}}$, we see that for all $g \in \text{supp}(g)$ and $\underline{\nu} \in \bar{U}$, $f(g)\pi(g)\underline{\nu} \in \frac{1}{\text{vol}(\text{supp}(f))}U$. It follows that $\pi(f)\underline{\nu} \in U$.

Rest proved similarly.

Corollary 32. Let $\{f_{n}\} \subset C_c(G)$ be an approximate identity. Then $\pi(f_{n})\underline{\nu} \to \underline{\nu}$.

Example 33 (Smoothing). Let $V \subset L^{2}(G)$ be a closed $G$-invariant subspace. Then $V \cap C(G)$ is dense in $G$.

Proof. It suffices to show that $\pi(f)\varphi \in C(G)$ for each $f \in C_c(G)$, $\varphi \in L^{2}(G)$. Indeed,

\[
\pi(f)\varphi(x) = \int f(g)\varphi(g^{-1}x)\,dg = \int f(xg)\varphi(g^{-1})\,dg
\]

so that

\[
|\pi(f)\varphi(x) - \pi(f)\varphi(y)| \leq \int |\delta(g)(f(xg^{-1}) - f(yg^{-1}))\varphi(g)|\,dg
\]

\[
\to 0 \quad \text{as} \quad y \to x
\]

since $f$ is uniformly continuous and $\delta$ is bounded on any compact set.

Suppose now that $G$ is compact.

Theorem 34 (Peter–Weyl I). We have

\[
L^{2}(G) = \bigoplus_{\pi} C(\pi).
\]

Proof. Let $V = (\bigoplus_{\pi} C(\pi))^\perp$ and note that $V$ is a subrepresentation of $(L^{2}(G),\mathcal{R})$. If $V \neq \{0\}$ let $f \in V$ be non-zero, and by continuity of the $G$-action on $L^{2}(G)$ let $U \subset G$ be a symmetric, conjugation-invariant neighbourhood of 1 such that $\|R_{u}f - f\|_{2} \leq \frac{1}{2}\|f\|$. Let $\chi \in C_{c}(U)$ be positive, satisfy $\chi(u) = \chi(u^{-1})$, integrate to 1 and be conjugation invariant. Then $\|R(\chi)f - f\|_{2} \leq \frac{1}{2}\|f\|$ and in particular $R(\chi) : V \to V$ is a non-zero operator. It is also self-adjoint and compact. By the spectral theorem its eigenspaces are finite-dimensional and it follows that $V$ contains $G$-finite vectors, a contradiction.

Corollary 35 (Peter–Weyl II). $\bigoplus_{\pi} C_{c}(\pi)$ is dense in $C(G)$.

Proof. Since the matrix coeff of the tensor product are products, this is a subalgebra closed under complex conjugation and it suffices to show it separates the points. By $G$-invariance it suffices to separates points from the identity.
For this consider \( \bigcap \ker \pi \). Every \( f \in L^2(G) \) is invariant by this closed subgroup, so it’s trivial. It follows that for any \( g \in G \) there is \( \pi \) such that \( \pi(g) \neq \text{id} \). Let \( \psi \in V_\pi \) be of norm 1 such that \( \pi(g)\psi \neq \psi \). Then by unitarity \( \langle \pi(g)\psi, \psi \rangle \neq 1 \) and hence
\[
\Phi_{\psi, \psi}(g) \neq 1.
\]

**Theorem 36 (Peter–Weyl II).** Every irrep of \( G \) is finite-dimensional; for any representation \( V_K \) is dense in \( V \).

**Proof.** Clearly the second assertion implies the first. We first note that the argument of Theorem 34 shows that \( \{ \pi(\chi)\psi \mid \psi \in V, \chi \in C(G) \} \) is dense in \( V \). Now to see that \( V_K \) is dense in \( V \) it suffices to show that \( V_K = \{ \pi(f)\psi \mid \psi \in V, f \in C(G)_G \} \) is dense. For that note that for any neighbourhood \( W \subset V \) of zero, if \( f \) is \( \epsilon \)-close to \( \chi \) for \( \epsilon \) small enough then \( \pi(f - \chi)\psi \in W \). □
CHAPTER 2

Lie Groups and Lie Algebras

2.1. Smooth manifolds

2.1.1. Manifolds.

Definition 37. Let \( U \subset \mathbb{R}^n \) be open. Then \( C^\infty(U; \mathbb{R}^m) \) is the set of infinitely differentiable \( \mathbb{R}^m \)-valued functions on \( U \).

Definition 38. A coordinate chart (or patch) in a topological space \( M \) is a pair \((U, \varphi)\) where \( U \subset M \) is open and \( \varphi: U \to \mathbb{R}^n \) is a homeomorphism onto an open subset of \( \mathbb{R}^n \). Two coordinate patches \((U_1, \varphi_1), (U_2, \varphi_2)\) are compatible if \( \varphi_1 \mid_{U_1 \cap U_2} \circ (\varphi_2 \mid_{U_1 \cap U_2})^{-1} \) is a smooth map.

An atlas on \( M \) is a covering of \( M \) by compatible coordinate patches. A smooth manifold is a pair \((M, \mathcal{A})\) where \( M \) is a second countable topological space and \( \mathcal{A} \) is an atlas on \( M \).

Example 39. \( \mathbb{R}^n, S^n, T^n \).

Lemma 40. If two charts are compatible with an atlas they are compatible with each other.

Corollary 41. Every atlas is contained in a maximal atlas, namely the set of all charts compatible with the given atlas.

Definition 42. A maximal atlas is also known as a smooth structure on \( M \).

Example 43. Exotic spheres.

Lemma 44. If \( m \neq n \) \( \mathbb{R}^m, \mathbb{R}^n \) are not locally homeomorphic so for a connected manifold the dimension need not be assumed constant.

Definition 45. Let \( M, N \) be smooth manifolds. A map \( f: M^m \to N^n \) is smooth if for every charts \((U, \varphi)\) of \( M \) and \((V, \psi)\) of \( N \), \( \psi \circ f \circ \varphi^{-1} \) is smooth.

Lemma 46. Composition of smooth maps is smooth.

2.1.2. Tangent and cotangent spaces. Fix a vector space \( k \).

Definition 47. A Lie algebra over \( k \) is a \( k \)-vector space \( g \) together with a bilinear form \([\cdot, \cdot]: g \times g \to g\) satisfying:

1. (alternating) \([X, X] = 0\)
2. (Jacobi identity) \([[[X, Y], Z] + [[Y, Z], X] + [[[Z, X], Y] = 0\).

Example 48 (Standard constructions). Let \( A \) be an associative \( k \)-algebra. We get two natural Lie algebras from it:

1. \( A \) itself, equipped with \([a, b] = ab - ba\).
2. Call \( d \in \text{End}_{k, vsp}(A) \) a derivation if \( d(ab) = d(a)b + ad(b) \). Then the space \( \mathcal{D}_A \) of derivations is a Lie subalgebra of \( \text{End}_{k, vsp}(A) \).
(3) One canonical example: \( A = C^\infty(M) \); then \( \mathcal{D}_M \stackrel{\text{def}}{=} \mathcal{D}_{C^\infty(M)} \) is called the set of (smooth) vector fields on \( M \).

**Lemma 49** (Localization of vector fields). Let \( X \in \mathcal{D}_M, f, g \in C^\infty(M) \).

1. Let \( f \) be constant. Then \( Xf \equiv 0 \).
2. Let \( f(p) = 0 \). Then \( (Xf^2)(p) = 0 \).
3. Let \( f \) be constant in a neighbourhood of \( p \). Then \( (Xf)(p) = 0 \). In particular, if \( f = g \) in a neighbourhood of \( p \) then \( X(f)(p) = Xg(p) \).

**Proof.** Say \( f(p) = 1 \) for all \( x \). Then \( Xf = X(f^2) = 2f \cdot Xf = 2Xf \). It follows that \( Xf \equiv 0 \).

Let \( U \) be a neighbourhood of \( p \in U \) and suppose \( f \upharpoonright_U \equiv 1 \). Choose \( g \in C^\infty(U) \) such that \( g(p) \neq 0 \). Since \( fg = g \) we have \( Xf \cdot g + f \cdot Xg = Xg \), Evaluating at \( p \) we get \( Xf(p)g(p) = 0 \) so \( Xf(p) = 0 \).

**Lemma-Definition 50.** \( I_p = \{ f \in C^\infty(M) \mid f(p) = 0 \} \) is a maximal ideal of \( C^\infty(M) \).

**Lemma-Definition 51** (Hadamard). The contangent space \( T^*_p M = I_p/I^2_p \) is a vector space of dimension \( n \) and \( \bigcup_{p \in M} T^*_p M \) is a vector bundle, the contangent bundle.

**Proof.** Let \( f \) vanish in a neighbourhood \( U \) of \( p \), and let \( g \in C^\infty(U) \) vanish at \( p \) as well. Then \( f = fg \in I^2_p \). It follows that \( f, g \in C^\infty(M) \) agree in a neighbourhood of \( p \) then \( f - g \in I^2_p \). We can now work locally, in particular near \( \emptyset \in \mathbb{R}^n \). We next show that every class in \( I_p/I^2_p \) has a linear representative. Indeed let \( f \) be smooth in a neighbourhood of \( \emptyset \in \mathbb{R}^n \) and set \( g(t) = f(tx) \). Then

\[
\begin{align*}
  f(x) - f(\emptyset) &= g(t) - g(1) = \int_0^1 g(t) \, dt \\
  &= \int_0^1 x \cdot \nabla f(tx) \, dt \\
  &= \sum_{i=1}^n x_i \cdot \int_0^1 \frac{\partial}{\partial x_i} f(tx) \, dt \\
  &= \nabla f(\emptyset) \cdot x + \sum_{i=1}^n x_i h_i
\end{align*}
\]

where \( h_i(\emptyset) = \int_0^1 \frac{\partial}{\partial x_i} f(tx) \, dt - \frac{\partial f}{\partial x_i}(\emptyset) \in I_0 \). It follows that \( f(x) - f(\emptyset) = \nabla f(\emptyset) \cdot x + \sum_{i=1}^n x_i h_i \). To see that the linear functions inject into \( I_p/I^2_p \) (so that the dimension is \( n \) note that each linear function has a non-zero directional derivative, but that operation is a derivation in \( C^\infty(U) \) \( (U \subset \mathbb{R}^n) \) and vanishes on elements of \( I^2_p \). \)

**Lemma-Definition 52** (The tangent space). The linear dual \( T_p^* M = \text{Hom}_\mathbb{R}(T^*_p M, \mathbb{R}) \) is called the tangent space. The resulting bundle is called the tangent bundle.

1. The pairing \( (X, f) \mapsto Xf(p) \) associates to each vector field \( X \) a linear functional on \( T^*_p X \).
2. The resulting map \( \mathcal{D}_M \to \left( T^*_p M \right)' \) is surjective.

**Conclusion 53.** In local coordinates, a vector field is an operator of the form \( \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x^i} \).
EXERCISE 54. $T_pM$ is also the space of derivations on the algebra of germs of smooth functions at $p$.

PROPOSITION 55 (Canonical sheaf). (1) Let $X$ be a vector field on $M$, $U \subset M$ an open set. For $f \in C^\infty(U)$ and $p \in U$ let $h \in C^\infty_c(U)$ such that $h \equiv 1$ near $p$ and set $(X |_U f)(p) = (X(hf))(p)$ (note that $hf \in C^\infty_c(M)$). Then $X |_U$ is a well-defined vector field on $U$ and $X \mapsto X |_U$ is a map of lie algebras.

(2) (Patching) Let $\{U_i\}_{i \in I}$ be an open cover of $M$. Let $X, Y$ be a vector fields on $M$ and suppose that $X |_{U_i} = Y |_{U_i}$ for all $i$ then $X = Y$.

(3) (Gluing) Let $\{U_i\}_{i \in I}$ be an open cover of $M$ and suppose given for each $i$ a vector field $X_i$ on $U_i$ such that $X_i |_{U_i \cap U_j} = X_j |_{U_i \cap U_j}$ for all $i, j$. Then there is a vector field $X$ on $M$ such that $X_i = X |_{U_i}$.

2.1.3. Derivatives of maps.

LEMMA-DEFINITION 56. Let $\varphi: M \to N$ be a smooth map. Let $p \in M$ and $v \in T_pM$. Then the map $d\varphi_p(v): C^\infty(N) \to \mathbb{R}$ given by $f \mapsto v(f \circ \varphi)$ is a local derivation at $\varphi(p)$. It is called the differential of $\varphi$. The map $d\varphi_p: T_pM \to T_{\varphi(p)}N$ is linear and extends to a smooth map $d\varphi: TM \to TN$ compatible with $\varphi$. The construction is functorial (in other words, the chain rule holds).

THEOREM 57 (Inverse and implicit function theorems). Let $\varphi: M \to N$ be smooth.

(1) Suppose $d\varphi_p$ is injective. Then $\varphi$ is injective in a neighbourhood of $p$.

(2) Suppose $d\varphi_p$ is a surjective. Then $\varphi$ is an open map in a neighbourhood of $p$.

(3) Suppose $d\varphi_p$ is an isomorphism. There are open neighbourhoods of $p$ and $\varphi(p)$ for which $\varphi$ is a diffeomorphism.

(4) Suppose $d\varphi_p$ is surjective for $p$ on a level set $P = \varphi^{-1}(n)$. Then the level set is a submanifold of dimension $\dim N - \dim M$.

DEFINITION 58. A smooth map $f: M \to N$ is a:

(1) Submersion if $df_p$ is injective for every $p \in M$.

(2) Local embedding if it is a submersion and for every $p \in U \subset M$ there is $f(p) \in V \subset N$ such that $f |_{f^{-1}(V)}$ is a homeomorphism onto its image (with the relative topology).

(3) An embedding if it is an injective immersion which is a homeomorphism onto its image.

(4) A diffeomorphism if it has a smooth inverse.

DEFINITION 59. A parametrized submanifold of $N$ is a pair $(M, f)$ where $f: M \to N$ is an injective submersion. Two parametrizations $(M_1, f_1), (M_2, f_2)$ are equivalent if they are conjugate by a diffeomorphism of $M_1, M_2$. A submanifold of $N$ is an equivalence class.

If $(M, f)$ is a parametrized submanifold $N$ then $T(N) = f_* (TM)$ is a subbundle of $TN |_M$. It is independent of the choice of parametrization. Conversely, we’d like to investigate when a choice of subspace of $T_pM$ at each $p$ corresponds to a submanifold.

DEFINITION 60. Let $\gamma: [a, b] \to M$ be a smooth curve. We then set $\dot{\gamma}(t) = d\gamma\left(\frac{\partial}{\partial t}\right)(t)$. We say $\gamma$ is an integral curve of $X \in \mathcal{D}_M$ if $\dot{\gamma}(t) = X(\gamma(t))$ for each $t$.

- The Picard Theorem on ODE shows that for any $X$ and $p \in M$ there is an integral curve of $X$ through $p$ living on an interval about 0, and that any two integral curves with $\gamma(0) = p$ agree on their interval of definition.
We now generalize this from 1-dimensional submanifolds to higher dimension.

**Definition 61.** A *distribution* of dimension \( k \) on \( M \) is equivalently either of:

1. A smooth choice of \( k \)-dimensional subspaces \( V_p \subset T_p M \) for each \( p \in M \).
2. A smooth section of the Grassmanian bundle, or a subbundle of \( TM \).
3. For a covering set of neighbourhoods \( U \subset M \) choices of vector fields \( \{X_i\}_{i=1}^k \subset \mathcal{D}_U \) so that for each \( p \in U \), \( \{X_i(p)\} \subset T_p M \) are linearly independent and so that \( V_p = \operatorname{Span}_\mathbb{R} \{X_i(p)\}_i \) is independent of \( U \) as long as \( p \in U \).

Call a vector field \( X \in \mathcal{D}_M \) a *section* of the distribution \( V \) if (1) \( X_p \in V_p \) for each \( p \) iff (2) it is a section of the subbundle (3). For each \( U \) there are \( a_i \in C^\infty(U) \) so that \( X|_U = \sum_i a_i X_i \).

**Definition 62.** Call a submanifold \( (N^k, \varphi) \) of \( M \) tangent to the distribution \( V \) if for each \( p \in N \), \( d\varphi_p \) is an isomorphism of \( T_p N \) and \( V_{\varphi(p)} \subset T_{\varphi(p)} M \).

**Observation 63.** Suppose \( N \subset M \) is tangent to \( V \), and let \( X, Y \) be sections of \( V \). We can then think of \( X, Y \) as vector fields on \( N \), so that \([X,Y] \) is a vector field on \( N \) as well. It follows that \([X,Y] \) is also a section of \( V \).

In fact, this necessary condition is also sufficient:

**Theorem 64** (Frobenius). The following are equivalent for a distribution \( V \) on \( M \):

1. Through each \( p \in M \) there is a unique (up to equivalence) submanifold tangent to \( V \); this submanifold is injectively submersed.
2. The distribution is completely integrable: for every two sections \( X, Y \) of \( V \), the vector field \([X,Y] \) is also a section.

**Remark 65.** In the local view above it suffices to check that the integrability condition on the spanning fields: \([X_i,X_j] = \sum_k a_k X_k \) for some \( a_k \in C^\infty(U) \).

### 2.2. Lie groups

**Definition 66.** A *Lie group* is a group object in the category of smooth manifolds, in other words a smooth manifold \( G \) together with smooth maps \( \cdot : G \times G \to G \) and \( ^{-1} : G \to G \) such that \((G, \cdot, ^{-1})\) is an abstract group. A *homomorphism* of Lie group is an abstract homomorphism which is also a smooth map.

**Example 67.** The basic example is \( \mathbb{R} \), but we also have:

1. \( \mathbb{R}^n, (\mathbb{R}/\mathbb{Z})^n = \mathbb{R}^n/\mathbb{Z}^n \)
2. \( \text{GL}_n(\mathbb{R}), \text{SL}_n(\mathbb{R}), \text{GL}_n(\mathbb{C}), \text{Sp}_{2n}(\mathbb{R}) \)
3. \( \text{O}(n), \text{SO}(n), \text{SO}(Q) = \text{SO}(p,q), \text{U}(n), \text{SU}(n) \)
4. Direct and semidirect products.
5. \( \operatorname{Isom}(\mathbb{E}^n), \operatorname{Isom}(M,g) \)
6. \( \text{Aff}_n(\mathbb{R}) \)

**Definition 68.** An *action* of a Lie group \( G \) on a smooth manifold \( M \) is a smooth map \( \cdot : G \times M \to M \) which is a group action.

**Definition 69.** A *Lie subgroup* \( H \) of the Lie group \( G \) is a subgroup \( H < G \) which is also a submanifold, in other words the image of an injective immersion of Lie groups.
EXAMPLE 70. Lie of irrational slope on a torus.

REMARK 71. There is some play in the joints here.

(1) Enough to assume $C^2$, and may assume real-analytic (any $C^2$ structure is compatible with a unique smooth, even real-analytic, structure).

(2) Sophus Lie actually considered local Lie group actions.

2.3. Lie Algebras and the exponential map

2.3.1. Lie algebra. The Lie group $G$ acts on itself by left multiplication. This regular action is a smooth action. In particular each $g \in G$ acts by translation on the set of vector fields of $G$, and we call a vector field $X$ left-invariant if $g \cdot X = X$. Recall that for any manifold we have a surjective map $\{D_M\} \to T_p M$.

L E M M A 72. Restricting this map to the left-invariant vector fields on $G$ gives a linear isomorphism $\{\text{left-invariant vector fields on } G\} \to T_e G$.

P ROOF. For the inverse map, for any manifold $M$ a smooth action of $G$ on $M$ extends to a smooth action on $TM$ by $g \cdot (p,v) = (gp, dg(v))$ where $dg$ is the derivative of the map $g : M \to M$. In particular, $G$ acts on $TG$. Now for $v \in T_e G$ the orbit $g \mapsto g \cdot (e,v)$ is a smooth left-invariant vector field. □

Note that if $X, Y$ are left-invariant so is $[X, Y]$.

D E F I N I T I O N 73. The Lie algebra of $G$ is the Lie algebra of left-invariant vector fields, equivalently the same Lie algebra realized as the tangent space $T_e G$. We write $\mathfrak{g} = \text{Lie}(G)$ for the Lie algebra.

T H E O R E M 74. If $f \in \text{Hom}(G, H)$ then $df : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

L E M M A 75. A connected Lie group is generated by any open subset

T H E O R E M 76. Every subalgebra exponentiates to a subgroup

P ROOF. The distribution defined by the subalgebra is integrable, so apply Frobenius. The leaf through the origin is self-invariant, hence a subgroup. □

2.3.2. Exponential map.

L E M M A-D E F I N I T I O N 77. The integral curves through left-invariant vector fields live forever. Write the integral curve through $X \in \mathfrak{g}$ as $t \mapsto \exp(tX)$.

R E M A R K 78. Uniqueness of integral curves shows that indeed this only depends on $tX \in \mathfrak{g}$ rather than on $t, X$ separately.

P R O P O S I T I O N 79. $\exp : \mathfrak{g} \to G$ is a local diffeomorphism with derivative $\text{Id}$.

P ROOF. Differentiate ODE and inverse function theorem □

L E M M A 80. Homomorphisms respect the exponential map

P ROOF. $f(\exp(tX))$ is an integral curve of $df(X)$ □

P R O P O S I T I O N 81. Exponential map of $\text{GL}_n(\mathbb{R})$ is given by the matrix exponential.
2.4. Closed Subgroups

**Theorem 82** (Cartan 1930). Let \( H < G \) be a closed subgroup. Then \( H \) is a Lie subgroup (in particular, a submanifold of \( G \)).

**Proof.** Let \( \mathfrak{h} = \{ X \in \mathfrak{g} \mid \forall t \in \mathbb{R} : \exp(tX) \in H \} \). Then \( \mathfrak{h} \subset \mathfrak{g} \) is closed under scaling. Also, \( Z(t) = \log(\exp(tX)\exp(tY)) \) has \( Z(0) = 0, Z'(0) = X + Y \) so \( Z(t) = t(X + Y) + o(t^2) \) and it follows that \( \exp(t(X + Y)) \in H \). Now \( \exp : \mathfrak{h} \rightarrow H \) is locally bijective (use local exponential coordinateS).

**Lemma 83.** Let \( V \subset G \) be a small enough neighbourhood of the identity. Then

**Theorem 84.** Let \( H \) be a closed connected subgroup of \( G \). Then \( G/H \) has a unique manifold structure such that \( \pi : G \rightarrow G/H \) is smooth. Furthermore, the regular action of \( G \) on \( G/H \) is a Lie group action.

**Theorem 85.** A map of Lie groups is a covering iff its derivative is an isomorphism.

**Proof.** A covering map is a local diffeo, hence gives isom of Lie algebras. Conversely, let \( df = df_e \) be an isomorphism for \( f : G \rightarrow H \). By homogeneity \( df_g \) is injective for each \( g \in G \) so \( f \) is a local diffeomorphism. The kernel \( \Gamma = \text{Ker}(f) \) is a closed subgroup, which is zero-dimensional hence discrete. Let \( U \subset G \) be a small enough neighbourhood so that its translates by \( \Gamma \) are disjoint and such that \( f \mid_U \) is a diffeo. Then \( f^{-1}(f(U)) \simeq \Gamma \times U \).

**Theorem 86.** Let \( df : \mathfrak{g} \rightarrow \mathfrak{h} \) be a Lie algebra homomorphism. If \( G \) is simply connected and \( H \) is connected then this lifts to \( f \).

**Proof.** Realize the graph of \( f \) as a subgroup of \( G \times H \) corresponding to a lie subalgebra. Projection on \( G \), so the graph is a function.

**Theorem 87** (Ado). Every finite-dimensional Lie algebra has a faithful representation into \( \mathfrak{gl}_n(\mathbb{R}) \).

**Proof.** Adjoint gives a representation mod centre.

**Corollary 88.** Every Lie algebra is the Lie algebra of some group.

**Theorem 89.** Let \( H < G \) be a closed subgroup. Then \( G/H \) has a unique manifold structure such that the quotient map is smooth.

**Proof.** Local exponential coordinates.

2.5. The adjoint representation

**Definition 90.** Let \( g \in G \). Then \( \text{Ad}_g : G \rightarrow G \) given by \( \text{Ad}_g(x) = gxg^{-1} \) is an automorphism, in particular a group homomorphism. We also write \( \text{Ad}_g \) for its derivative, \( \text{Ad}_g : \mathfrak{g} \rightarrow \mathfrak{g} \).

**Lemma 91.** \( \text{Ad} : G \rightarrow \text{GL}(\mathfrak{g}) \) is a smooth representation.

**Definition 92.** Write \( \text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}) \) for the derivative of the adjoint representation.

**Theorem 93.** \( \text{ad}_X \cdot Y = [X, Y] \).
PROOF. Since \( \text{ad} \) is the derivative of \( \text{Ad} \), \( \exp(t \text{ad}_X) = \text{Ad}_{\exp(tX)} \) in \( \text{GL}(g) \). It follows that
\[
\begin{align*}
\text{ad}_X \cdot Y &= \frac{d}{dt} |_{t=0} \text{Ad}_{\exp(tX)} Y \\
&= \frac{d}{dt} |_{s=0} \frac{d}{ds} |_{s=0} \text{Ad}_{\exp(tX)} \exp(sY) \\
&= \frac{d}{ds} |_{s=0} \frac{d}{dt} |_{t=0} \exp(tX) \exp(sY) \exp(-tX) \\
&= \frac{d}{ds} |_{s=0} (X_e \cdot (\exp(sY))_s - X_{\exp(sY)}) .
\end{align*}
\]
More precisely, this means that for \( f \in C^\infty(G) \),
\[
\langle df, \text{ad}_X \cdot Y \rangle = \frac{d}{ds} |_{s=0} \langle df_{\exp(sY)}, X_e \cdot (\exp(sY))_s - X_{\exp(sY)} \rangle .
\]
Now
\[
\frac{d}{ds} |_{s=0} \langle df_{\exp(sY)}, X_e \cdot (\exp(sY))_s \rangle = \frac{d}{ds} |_{s=0} \langle d(R_{\exp(sY)} f)_e, X_e \rangle = \left( \frac{d}{ds} |_{s=0} d(g \mapsto f(g \exp(sY)))_e, X_e \right) = (XY f)(e)
\]
and
\[
\frac{d}{ds} |_{s=0} \langle df_{\exp(sY)}, X_{\exp(sY)} \rangle = (YX f)(e)
\]
so we are done. \( \square \)

**COROLLARY 94.** \( \text{ad} : g \to \text{End}(g) \) is a Lie algebra representation: \( \text{ad}_{[X,Y]} = [\text{ad}_X, \text{ad}_Y] \).

**PROOF.** This follows immediately from the Jacobi identity. \( \square \)

**COROLLARY 95.** Let \( H < G \) be connected Lie groups. Then \( H \) is normal iff \( \mathfrak{h} \) is a Lie ideal.

**PROOF.** If \( H \) is normal then \( H \) is \( \text{Ad} \)-stable hence \( \mathfrak{h} \) is \( \text{Ad} \)-stable hence \( \mathfrak{h} \) is ad-stable. Conversely, for \( X \) close enough to the origin we have \( \exp(\text{ad}_X) = \sum_{k=0}^\infty \frac{1}{k!} (\text{ad}_X)^k \). Now if \( \mathfrak{h} \) is \( \text{ad} \)-stable it follows that it is also \( \exp(\text{ad}_X) \)-stable and hence \( \text{Ad}_{\exp X} \)-stable. But by the group-algebra correspondence this means \( H \) is \( \text{Ad}_{\exp X} \)-stable. Since the small \( X \) generate \( G \) we are done. \( \square \)

**COROLLARY 96.** Let \( G \) be connected. Then \( Z(G) = \ker (\text{Ad} : G \to \text{GL}(g)) \).

**PROOF.** \( g \in G \) is central iff for all small enough \( X \), \( g \exp X g^{-1} = \exp X \) iff \( \exp(\text{Ad}_g X) = \exp X \) iff \( \text{Ad}_g X = X \).

**COROLLARY 97.** Let \( G \) be connected. Then \( g \) is abelian iff \( G \) is abelian iff \( \exp : g \to G \) is a surjective group homomorphism.

**PROOF.** If \( \text{ad}_X = 0 \) for all \( X \) then \( \exp(\text{ad}_X) = \text{Id} \) for all \( X \) so a neighbourhood of the identity is contained in \( \ker(\text{Ad}) \). If \( G \) is abelian let \( X, Y \in g \). Then \( t \mapsto \exp(tX) \exp(tY) \) is a group homomorphism \( \mathbb{R} \to G \). Since its derivative at \( t = 0 \) is \( X + Y \) we conclude that \( \exp(tX) \exp(tY) = \exp(t(X + Y)) \). Now setting \( t = 1 \) shows that \( \exp \) is a homomorphism, and since the image contains a generating set it’s surjective. Finally, if \( \exp \) is a surjective homomorphism then its image \( G \) is abelian. \( \square \)

**THEOREM 98.** A connected abelian Lie group is of the form \( \mathbb{R}^n \times \mathbb{T}^m \).

**PROOF.** \( \ker(\exp) \) is a discrete subgroup of \( \mathbb{R}^d \). \( \square \)
CHAPTER 3

Compact Lie groups

3.1. Linearity

As an application of our representation theory of compact groups we get:

**Theorem 99.** Every compact Lie group has a faithful finite-dimensional representations. Equivalently, every compact group is isomorphic to a closed subgroup of some $U(n)$.

**Proof.** The representation of $G$ on $L^2(G)$ is faithful. By Peter–Weyl it follows that $\cap_{\pi \in \hat{G}} \text{Ker}(\pi) = \{e\}$. Let $\square$

3.2. Characters and cocharacters of tori

Let $\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m$ be tori. We’d like to study $\text{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$. The cases $m = 1$ (characters) and $m = n$ (automorphisms) are particularly important.

First, let $f : \mathbb{Z}^n \to \mathbb{Z}^m$ be a group homomorphism. Extending scalars gives a homomorphism $f_\mathbb{R} = f \otimes \mathbb{R} : \mathbb{Z}^n \otimes \mathbb{R} \to \mathbb{Z}^m \otimes \mathbb{R}$. Since $f_\mathbb{R}(\mathbb{Z}^n) = f(\mathbb{Z}^n) \subset \mathbb{Z}^m$, $f_\mathbb{R}$ descends to a homomorphism $\bar{f} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^m/\mathbb{Z}^m$.

**Lemma 100.** The map $f \mapsto \bar{f}$ is an isomorphism $\text{Hom}(\mathbb{Z}^n, \mathbb{Z}^m) \to \text{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$.

**Proof.** We need to construct the inverse map. For this let $\exp_n : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the quotient map, which is also the exponential map of this commutative Lie group with kernel $\mathbb{Z}^n$. Then given $\bar{f} \in \text{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$ consider the linear map $d\bar{f} \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^m)$. The usual identity $\bar{f}(\exp_n X) = \exp_m (d\bar{f}(X))$ here reads $\exp_m \circ d\bar{f} = \bar{f} \circ \exp_n$, in other words that $d\bar{f}(\mathbb{Z}^n) \subset \mathbb{Z}^m$ and hence that $f = d\bar{f} \mid_{\mathbb{Z}^n}$ is the desired element of $\text{Hom}(\mathbb{Z}^n, \mathbb{Z}^m)$. $\square$

**Corollary 101.** $\text{Aut}(\mathbb{T}^n) \simeq M_n(\mathbb{Z})^\times = \text{GL}_n(\mathbb{Z})$. In particular, $\text{Aut}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^m/\mathbb{Z}^m)$ is discrete.

**Corollary 102.** $\hat{\mathbb{T}}^n = \text{Hom}(\mathbb{T}^n, \mathbb{S}^1) = \{e(k \cdot x)\}_{k \in \hat{\mathbb{Z}}^n}$ where $\hat{\mathbb{Z}}^n = \text{Hom}(\mathbb{Z}^n, \mathbb{Z})$ is the dual lattice, and $e(z) = e^{2\pi iz}$.

**Proof.** $z \mapsto e(z)$ is an isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{S}^1$. $\square$

**Lemma 103.** Tori are topologically generated by single elements.

**Proof.** Let $\{1\} \cup \{\xi_i\}_{i=1}^n \subset \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Then $\xi$ is such an element. In fact (Weyl equidistribution) every orbit $\{x + j\xi\}_{j=1}^\infty$ is equidistributed in the torus. $\square$
3.3. The exponential map

From now on let \( G \) be a compact connected Lie group, \( g \) its Lie algebra, and let \( \text{Ad}: \ G \to \text{GL}(g) \) be the adjoint representation. Since \( G \) is compact we may fix a \( G \)-invariant inner product (and associated Euclidean norm) on \( g \).

**Lemma 104.** A connected compact Lie group has a bi-invariant Riemannian metric

**Remark 105.** The map \( g \mapsto g^{-1} \) is an isometry of this metric. In other words, we have a symmetric space. (c.f. PS ??)

**Proposition 106.** Fix a bi-invariant metric on \( G \). Then the Riemannian and Lie exponential maps agree.

**Proof.** Let \( \gamma(t) \) be a Riemannian geodesic based at the origin. Then \( t \mapsto \gamma(t_0 + t) \), \( t \mapsto \gamma(t) \gamma(t_0) \) and \( t \mapsto \gamma(t) \gamma(t_0) \) are also geodesics (because the group acts by isometries) which meet at \( t = 0 \) and have the same derivative at that time. It follows that \( \gamma(t_0 + t) = \gamma(t_0) \gamma(t) \), that is that the geodesic is a one-parameter subgroup.

**Corollary 107.** The exponential map of a connected compact Lie group is surjective.

**Corollary 108.** The intersection of two connected subgroups is connected.

**Proof.** The Lie algebra of the intersection is the intersection of the Lie algebras.

3.4. Maximal Tori

Fix a compact connected Lie group \( G \). A torus in \( G \) is a subgroup \( T \) of \( G \) isomorphic to \( \mathbb{T}^n \) for some \( n \). Being compact tori are always closed.

**Lemma 109.** Every \( g \in G \) is contained in a torus.

**Proof.** Suppose \( g = \exp(X) \) for \( X \in g \). Then \( \{ \exp(tX) \}_{t \in \mathbb{R}} \) is a connected commutative subgroup of \( G \). Its closure is a connected commutative compact group, that is a torus.

**Lemma 110.** Let \( T \) be a torus in \( G \), and let \( t \) be its Lie algebra. Then:

1. \( Z_G(T) \) is connected.
2. \( Z_G(t) = Z_G(T) \)
3. \( \text{Lie} Z_G(T) = Z_G(t) \).
4. \( N_G(T)^\circ = Z_G(T) \).

**Proof.**

1. Let \( t \in T \) generate a dense subgroup, so that \( Z_G(t) = Z_G(T) \). Embed \( G \) in \( U(n) \). Wlog the image of \( t \) is diagonal and then \( Z_{U(n)}(t) \) is block-diagonal, in particular connected. It follows that \( Z_G(T) = Z_G(t) = G \cap Z_{U(n)}(t) \) is connected.
2. If \( g \in Z_G(T) \) then \( \text{Ad}_g \in \text{Aut}(T) \) being trivial means that \( \text{Ad}_g \in \text{Aut}(t) \) is trivial. Conversely, the exponential map of \( T \) is surjective and for any \( H \in t \) and \( g \in Z_G(t) \) we have \( \text{Ad}_g(\exp H) = \exp(\text{Ad}_g H) = \exp H \).
3. If \( X \in Z_G(t) \) then for any \( s \in \mathbb{R} \), \( \text{Ad}_{\exp(sX)}|_t = \exp(\text{ad}_s X|_t) = \exp(0) = \text{Id} \) and hence \( \exp(sX) \in Z_G(t) \) and \( X \in \text{Lie} Z_G(t) \). Conversely, suppose that \( \text{Ad}_{\exp(sX)} \in Z_G(t) \) for all \( s \). Differentiating with respect to \( s \) we get that \( \text{ad} X|_t = 0 \) that is \( X \in Z_G(t) \).
Finally, let $N_G(T)$ act on $T$ by conjugation. This gives a continuous homomorphism $N_G(T) \to \text{Aut}(T) \simeq \text{GL}_r(\mathbb{Z})$. Since the latter group is discrete, the connected component is in the kernel and hence $N_G(T)^\circ \subset Z_G(T)$. Since $Z_G(T) \subset N_G(T)$ is connected we also have the reverse inclusion. □

### 3.4.1. Maximal tori

**Definition 111.** A maximal torus in $G$ is a torus in $G$, maximal wrt inclusion.

**Lemma 112.** Every element $g \in G$ is contained in a torus.

**Proof.** Suppose $g = \exp(X)$ for $x \in \mathfrak{g}$. Then the closure of $\{\exp(tX)\}_{t \in \mathbb{R}}$ is an abelian subgroup, hence a torus. □

**Corollary 113.** Every element of $g$ is contained in a maximal torus.

Fix a maximal torus $T$.

**Corollary 114.** $N_G(T)^\circ = Z_G(T) = T$.

**Proof.** Let $g \in Z_G(T)$ not belong to $T$. Then there is a torus $S \subset Z_G(T)$ such that $g \in S$. Then $ST$ is a torus properly containing $T$. □

**Definition 115.** The Weyl group of $G$ is $W(G : T) \overset{\text{def}}{=} N_G(T)/Z_G(T) = N_G(T)/T$.

**Theorem 116.** All maximal tori of $G$ are conjugate.

**Proof.** Let $S, T$ be maximal tori and let $X \in \text{Lie}S$, $Y \in \text{Lie}T$ be generic elements (that is, $\exp X, \exp Y$ are topological generators of the respective groups). Equip $\mathfrak{g} = \text{Lie}G$ with a $G$-invariant inner product, and let $g \in G$ minimize

$$f(g) = \|\text{Ad}(g)X - Y\|^2.$$ Expressing $f$ as:

$$f(g) = \|\text{Ad}(g)X\|^2 + \|Y\|^2 - 2\langle \text{Ad}(g)X, Y \rangle$$

we see that we are minimizing $\langle \text{Ad}(g)X, Y \rangle$. Suppose the minimum is at $g_0$, and consider the derivative there. For every $Z \in \mathfrak{g}$ the derivative in the direction $Z$ is:

$$\langle \text{ad}Z \cdot (\text{Ad}(g_0)X), Y \rangle.$$ Letting $X_0 = \text{Ad}(g_0)X$ we see that

$$0 = \langle \text{ad}Z \cdot (\text{Ad}(g_0)X), Y \rangle$$

$$= \langle [Z, X_0], Y \rangle = -\langle [X_0, Z], Y \rangle$$

$$= -\langle \text{ad}X_0 \cdot Z, Y \rangle$$

$$= \langle Z, \text{ad}X_0 \cdot Y \rangle = \langle Z, [X_0, Y] \rangle$$

where we use that in every unitary representation $\pi$, $d\pi(X)$ is anti-hermitian. Since $Z$ is arbitrary, we see that $[X_0, Y] = 0$. This means that $X_0 \in Z_\mathbb{R}(Y) = t$. But since $X_0$ is generic for $g_0Sg_0^{-1}$ we conclude that $g_0Sg_0^{-1} = T$. □
Corollary 117. \( T/W = G/\text{Ad}(G) \).

Proof. Fix a maximal torus \( T \). Every \( g \in G \) generates an abelian subgroup, hence contained in a maximal abelian subgroup, which is conjugate to \( T \). It follows that every conjugacy class has a representative in \( T \), so \( T/W \) surjects on the set of conjugacy classes. Conversely, let \( t, t' \in T \) be conjugate in \( G \).

3.4.2. Example: three-dimensional groups. Let \( G = \text{SU}(2) \) act on \( \mathbb{C}^2 \). The action on \( S^3 \) is simply transitive, so \( \text{SU}(2) \cong S^3 \); in particular it is simply connected. Now \( Z(\text{SU}(2)) = \{ \pm I \} \), so the groups it covers are \( \text{SU}(2) \) and its image by the adjoint representation.

Lemma 118. The maximal tori are the maximal subalgebras of \( \text{SO}(3) \).

Proof. Let \( t = \text{Span} \left( \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right) \). Then the action of \( t \) on its orthogonal complement in \( \mathfrak{so}(3) \) is irreducible. □

Proposition 119. Let \( G \) be a three-dimensional connected compact Lie group. Then \( G \) covers \( \text{SO}(3) \).

Proof. Consider the adjoint representation \( \text{Ad}: G \to \text{GL}(\mathfrak{g}) \). Choosing a \( G \)-invariant inner product on \( \mathfrak{g} \), the image lies in \( \text{O}(3) \), in fact in \( \text{SO}(3) \) since it is connected. We consider the Lie algebra of the image. The group \( G \) is commutative, so \( \hat{T} = \text{Hom}_{\text{cts}}(T, S^1) \) and \( V_\chi = \{ v \in V \mid \pi(t)v = \chi(t)v \} \). We call \( \{ \chi \in \hat{T} \mid V_\chi \neq \{0\} \} \) the exponential weights of \( V \), \( V_\chi \) the weight spaces.

We show that the image is three-dimensional, so it equals \( \text{SO}(3) \) exactly. Can’t be a torus (then \( \mathfrak{g}/Z_\mathfrak{g} \) is cyclic) and can’t be trivial (\( \mathfrak{g} \) is noncommutative). □

3.5. Roots and weights

3.5.1. Weights. Let \( T \) be a torus. Let \( (\pi, V) \) be a finite-dimensional representation of \( T \) on a complex vector space. By the theory for general compact groups we have a direct sum decomposition

\[ V = \bigoplus_{\chi \in \hat{T}} V_\chi. \]

Since \( T \) is commutative, \( \hat{T} = \text{Hom}_{\text{cts}}(T, S^1) \) and \( V_\chi = \{ v \in V \mid \pi(t)v = \chi(t)v \} \). We call \( \{ \chi \in \hat{T} \mid V_\chi \neq \{0\} \} \) the exponential weights of \( V \), \( V_\chi \) the weight spaces.

We now find an alternative parametrization of \( \hat{T} \). For this let \( t \) be the Lie algebra, \( \exp: t \to T \) the exponential map. We have seen that \( \exp \) is also the universal covering map of \( T \); we write \( \Lambda \) for its kernel and call it the integral lattice.

Identify the Lie algebra of \( S^1 \) with \( \mathbb{R} \) so that the exponential map is \( e(z) = e^{2\pi iz} \). For a character \( \chi \in \hat{T} \) write \( \alpha = d\chi \in t^* = \text{Hom}(t, \mathbb{R}) \) for its derivative, giving the following commutative diagram:

\[
\begin{array}{ccc}
t & \xrightarrow{\exp} & T \\
\downarrow{\alpha} & & \downarrow{\chi} \\
\mathbb{R} & \xrightarrow{e} & S^1 \\
\end{array}
\]

Now \( \chi \circ \exp \) vanishes on \( \Lambda \), and it follows that \( \alpha(\Lambda) \subset \ker(e) = \mathbb{Z} \). The converse is also clear, so
CONCLUSION 120. $\chi \in \hat{T}$ iff $\alpha \in \Lambda^* = \{v \in t^* \mid v(\Lambda) \subset \mathbb{Z}\} \simeq \text{Hom}(\Lambda, \mathbb{Z})$.

We call $\Lambda^*$ the weight lattice of $T$, and from now on we index weight spaces with the weights $\alpha \in \Lambda^*$ rather than the corresponding exponential weights $\chi_\alpha \in \text{Hom}(T, S^1)$. Explicitely given $\alpha \in \Lambda^*$ and $H \in t$ we have $\chi_\alpha(\exp H) = e^{2\pi i \alpha(H)}$.

3.5.2. Complexification. Suppose now that $T$ acts on a real vector space $V$. Since every non-trivial character of $T$ takes complex values, $V$ realizes no character of $T$, and we consider the complexification $V_C = \mathbb{C} \otimes_{\mathbb{R}} V$.

The complex conjugation operator $z \mapsto \bar{z}$ of $\mathbb{C}$ then extends to an operation $\nu \mapsto \bar{\nu}$ on $V_C$ (fixing the image of $V$ in $V_C$), and also $\text{End}_C(V_C)$ (fixing the image of $\text{End}_R(V)$ there).

EXERCISE 121. A $(\mathbb{C}$-linear) subspace $W \subset V_C$ is of the form $U_C$ for an $(\mathbb{R}$-linear) subspace $U \subset V$ iff $W = \bar{W}$.

The $T$-action on $V$ then extends to a $T$-action on $V_C$, so we may write $V_C = \bigoplus_{\alpha \in \Lambda^*} V_\alpha$. Then for any $H \in t$ and $\nu \in V_\alpha$ we have

$$\pi(\exp(H)) \cdot \nu = e^{2\pi i \alpha(H)} \nu.$$ 

Taking complex conjugates it follows that

$$\pi(\exp(H)) \cdot \bar{\nu} = e^{-2\pi i \alpha(H)} \bar{\nu},$$

in other words that $\bar{\nu} \in V_{-\alpha}$. We conclude that $\alpha \neq 0$ is a weight iff $-\alpha$ is a weight and that $V_\alpha = V_{-\alpha}$.

3.5.3. Roots. Let $G$ be a connected compact Lie group and fix a maximal torus $T \subset G$.

DEFINITION 122. The rank of $G$ is the integer $\text{rk} G = \dim T$. The semisimple rank of $G$ is the rank of $G/Z(G)$, in other words the integer $\dim T - \dim Z(G)$.

DEFINITION 123. The real roots of $G$ (with respect to $T$) are the non-zero weights of the adjoint action of $T$ on $\mathfrak{g}$. Write $\Phi = \Phi(G : T)$ for the set of roots.

The weight space $\mathfrak{g}_0$ corresponding to the weight 0 (that is, the space of $T$-fixed vectors) is self-conjugate, hence is the complexification of the space of $T$-fixed vectors in $\mathfrak{g}$. Since $Z_G(T) = T$ we see that this is exactly $t_C$ so we have

$$\mathfrak{g}_C = t_C \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

REMARK 124. We will now show that the structure of $\mathfrak{g}$ can be computed from this decomposition.

Let $H \in \text{liet}$, $X_\alpha \in \mathfrak{g}_\alpha$. We then have

$$\text{Ad}(\exp(tH)) \cdot X_\alpha = e^{2\pi i \alpha(H)} X_\alpha.$$

Differentiating with respect to $t$ we conclude that

$$\text{ad}_H \cdot X_\alpha = 2\pi i \alpha(H) X_\alpha.$$

In other words, $\mathfrak{g}_\alpha$ is a joint eigenspace of $\{\text{ad}_H\}_{H \in t}$ where the eigenvalue of $H$ is $2\pi i \alpha(H)$.
DEFINITION 125. Given a real root $\alpha$, the map $H \mapsto 2\pi i\alpha(H)$ will be called the associated complex root. We denote both by $\alpha$, but it should be clear from context which is intended. Note that the real root is an element of $t_{\mathbb{R}}^*$ while the latter is a purely imaginary element of $t_{\mathbb{C}}^*$. Generally the real roots are useful when studying representation theory and the “root system”. The complex roots are useful when studying structure theory, that is in computing commutators in $\mathfrak{g}$. Recall that we also have an associated exponential root $\chi_\alpha: T \to S^1$ such that $\text{Ad}_t \cdot X_\alpha = \chi_\alpha(t)X_\alpha$ whenever $t \in T$, $X_\alpha \in \mathfrak{g}_\alpha$.

LEMMA 126. For $\alpha, \beta \in \Lambda^*$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha + \beta}$.

PROOF. Let $H \in \text{liet}$, $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$. Then by the Jacobi identity (writing $\alpha$ for the complex root)

$$[H, [X_\alpha, X_\beta]] = -[X_\alpha, [X_\beta, H]] - [X_\beta, [H, X_\alpha]]$$

$$= -[X_\alpha, -\beta(H)X_\alpha] - [X_\beta, \alpha(H)X_\alpha]$$

$$= (\beta(H) + \alpha(H))(X_\alpha, X_\beta)$$

$$= (((\alpha + \beta)(H))(X_\alpha, X_\beta).$$

We are now ready to begin studying structure theory in earnest. The following argument is taken from [Brocker–tom Dieck, Prop xxxx]

THEOREM 127. If $\text{rk} G = 1$ then $G$ is either $\text{SO}(3)$ or $\text{SU}(2)$.

PROOF. We begin with two preliminary observations

1. Given $\beta \in \Phi$ let $X_\beta \in \mathfrak{g}_\beta$. Then $X_{-\beta} = \bar{X}_\beta \in \mathfrak{g}_{-\beta}$ and we may consider $H_\beta = [X_\beta, X_{-\beta}]$. If $H_\beta$ were zero span $\{X_\beta, X_{-\beta}\} \subset \mathfrak{g}_\mathbb{C}$ would be a two-dimensional commutative subalgebra of $\mathfrak{g}_\mathbb{C}$. Since this subspace is stable by complex conjugation it would be the complexification of a two-dimensional commutative subalgebra of $\text{liet}$, and such subalgebras don’t exist when $\text{rk} G = 1$. It follows that $H_\beta \neq 0$ in such circumstances. We also note that $H_\beta = [\bar{X}_\beta, X_{-\beta}] = [X_{-\beta}, X_\beta] = -H_\beta$. It follows that $H_\beta \in i\mathfrak{t}_{\mathbb{R}}$, and that $iH_\beta \in \mathfrak{t}$.

2. Fix a non-zero $H \in \mathfrak{t}$. Since $\mathfrak{t}$ is one-dimensional, every real root $\alpha$ is determined by the non-zero real number $\alpha(H)$, and we order the roots by these numbers.

Now let $\beta$ be the smallest positive root, choose $X_\beta$ and $X_{-\beta}$ as above and let

$$V = \mathbb{C}X_{-\beta} \oplus \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha.$$ 

We then have:

1. $V$ is $\text{ad}_{X_\beta}$-invariant, since $\text{ad}_{X_\beta} \cdot X_{-\beta} \subset \mathfrak{g}_0$, and for $\alpha \geq 0$ $\text{ad}_{X_\beta} \cdot \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha + \beta}$ and $\alpha + \beta \geq 0$.

2. $V$ is $\text{ad}_{X_{-\beta}}$-invariant, since $\text{ad}_{X_{-\beta}}(X_\beta) = 0$, $\text{ad}_{X_{-\beta}} \cdot \mathfrak{t}_\mathbb{C} \subset \mathbb{C}X_{-\beta}$ and for any $\alpha > 0$ we have $\alpha \geq \beta$ so $\text{ad}_{X_{-\beta}} \cdot \mathfrak{g}_\alpha \subset \mathfrak{g}_{\alpha - \beta}$ with $\alpha - \beta \geq 0$.

Now let $H_\beta = [X_\beta, X_{-\beta}]$ as above. Since the adjoint representation is a Lie algebra representation (Corollary 94), $\text{ad}_{H_\beta} = [\text{ad}_{X_\beta}, \text{ad}_{X_{-\beta}}]$ so $V$ is also stable by $\text{ad}_{H_\beta}$, since $\text{ad}_{H_\beta}$ is a commutator it
follows that $\text{Tr}_C \left( \text{ad}_{H_\beta} \mid V \right) = 0$. On the other hand, we can compute this trace via the eigenspace decomposition:

$$\text{Tr}_C \left( \text{ad}_{H_\beta} \mid V \right) = 2\pi i \beta(H_\beta) + 0 + \sum_{\alpha > 0} \dim_C g_\alpha \cdot 2\pi i \alpha(H_\beta).$$

Rearranging the terms we conclude that

$$(\dim_C g_\beta - 1) \beta(iH_\beta) + \sum_{\alpha > \beta} \dim_C g_\alpha \cdot \alpha(iH_\beta) = 0.$$ 

Now $iH_\beta \in t$ is a non-zero multiple of $H$. In particular either all the numbers $\beta(iH_\beta), \alpha(iH_\beta)$ are all positive or they are all negative. Also, the coefficients $(\dim_C g_\beta - 1), \dim_C g_\alpha$ are all non-negative. It follows that $\dim_C g_\beta = \dim_C g_{-\beta} = 1$ and that $\dim_C g_\alpha = 0$ if $\alpha > \beta$, in other words that $g_C = g_{-\beta} \oplus t_C \oplus g_\beta$ is three dimensional.

3.5.4. The algebraic Weyl group. Continuing with our general group $G$ and maximal torus $T$, let $\alpha \in \Phi$ and let $u_\alpha = \ker(\alpha)$, a codimension-1 subspace of $t$, $G_\alpha = Z_G(u_\alpha)$.

**Lemma 128.** $u_\alpha$ is the Lie algebra of the kernel of the exponential root $\chi_\alpha$. In particular, $\exp(u_\alpha)$ is a closed subgroup of $T$ of codimension 1.

**Remark 129.** That kernel need not be connected (for example, the kernel of the root of $\text{SU}(2)$ consists of the disconnected centre). We will later see that this kernel has at most two connected components.

**Proposition 130.** $G_\alpha$ is a connected subgroup of semisimple rank 1. Moreover:

1. $\dim_C g_\alpha = \dim_C g_{-\alpha} = 1$ and $\pm \alpha$ are the only roots proportional to $\alpha$.

2. $W(G_\alpha : T) \cong C_2$.

3. Let $s_\alpha \in W(G_\alpha : T) \subset W(G : T)$ be the non-trivial element. Then $s_\alpha \in \text{GL}(t)$ is a reflection in the hyperplane $u_\alpha$.

**Proof.** $G_\alpha$ centralizes the Lie algebra of a torus, so by Lemma 110 it is connected. Since $T \supset \exp(u_\alpha)$ is commutative, we see that $T \subset G_\alpha$ so that $T$ is a maximal torus there as well. By construction, $u_\alpha \subset Z_{\text{Lie}G_\alpha}$ so the semisimple rank is at most 1. It is not zero then $G_\alpha$ is non-commutative: its Lie algebra contains both $t$ and $\mathcal{R}(g_\alpha \oplus g_{-\alpha})$, and these subspace do not commute.

Set $\tilde{G}_\alpha = G_\alpha / \text{Ker} \chi_\alpha$, and let $\tilde{T} = T / \text{Ker} (\chi_\alpha)$, a maximal torus there. This is a group of rank 1, hence isomorphic to one of $\text{SU}(2), \text{SO}(3)$.

1. Let $\beta$ be a root proportional to $\alpha$. Then $\pm \beta(H) = 0$ for any $H \in u_\alpha$ and it follows that $\mathcal{R}(g_\beta \oplus g_{-\beta}) \subset \text{Lie}G_\alpha$ and hence that $g_\beta \subset \text{Lie}C_{G_\alpha}$. The direct sum over all these subspaces is disjoint from $u_\alpha$ so they all inject into $\text{Lie}C_{G_\alpha} / u_{\alpha C}$. Being the complexified Lie algebra of $\tilde{G}_\alpha$ it is three-dimensional and it follows that $\dim_C g_\alpha = \dim_C g_{-\alpha} = 1$ and that there are no other roots proportional to $\alpha$.

2. If $g \in G_\alpha$ normalizes $T$ then its image in $\tilde{G}_\alpha$ normalizes its maximal torus $\tilde{T}$. Conversely, if the image of $g$ normalizes $\tilde{T}$ then for any $t \in T$ we have $gtg^{-1} \in T \text{Ker}(\chi_\alpha) = T$ so $g$ normalizes $T$. It follows that the quotient map induces an isomorphism of the Weyl groups $W(G_\alpha : T) \simeq W(\tilde{G}_\alpha : \tilde{T}) \simeq C_2$. 

25
(3) Since \( u_\alpha \) is central in \( \text{Lie} G_\alpha \) it is fixed by any element of \( G_\alpha \). The non-trivial element of \( W(\tilde{G}_\alpha : \tilde{T}) \) acts by inversion on \( \tilde{T} \), so \( s_\alpha \) acts by inversion on \( t/\alpha \), that is by reflection in \( u_\alpha \) on \( t \).

\[ \Box \]

**Remark 131.** We call a root **reduced** if it is not a multiple of another root, and we see that here every root is reduced.

Since \( N_{G\alpha}(T) \subset N_G(T) \) we can think of \( s_\alpha \in N_{G\alpha}(T)/T \) as an element of \( W = N_G(T)/T \). This element is a reflection on \( t \) fixing \( u_\alpha \). Having equipped \( g \) with an inner product, the Weyl group acts by isometries on \( t \) so \( s_\alpha \) acts as the orthogonal reflection in \( u_\alpha \). We note that \( W \) also acts on the dual space \( t^* \) fixing the dual lattice \( \Lambda^* \) and the roots \( \Phi \) and that \( s_\alpha(\alpha) = -\alpha \).

**Definition 132.** Call \( s_\alpha \) the root reflection associated to the root \( \alpha \). The subgroup of the Weyl group generated by the root reflections will be called the algebraic Weyl group.

**Corollary 133.** Let \( 3 = Z(\mathfrak{g}) \) be the Lie algebra of the centre of \( G \) and let \( V = \{ v \in t^* \mid v(3) = 0 \} = (t/3)^* \). Then \( (V, \Phi) \) is a root system, in that it has the following properties:

1. \( \Phi \subset V \) is a finite set not containing \( \{0\} \).
2. \( \text{Span}_\mathbb{R} \Phi = V \).
3. For every \( \alpha \in \Phi \), the reflection \( s_\alpha \) in the hyperplane perpendicular to \( \alpha \) preserves \( \Phi \) setwise.

**Example 134.** Let \( G = \text{SU}(3) \). Let \( T = \{ \text{diag}(e(i\theta_1), e(i\theta_2), e(i\theta_3)) \mid \theta_1 + \theta_2 + \theta_3 = 0 \} \). This is a torus (isomorphic to \( (\text{SU}(1)^3)^2 \)). To see that it is maximal and compute its Weyl group, restrict the standard representation of \( \text{SU}(3) \) to \( \mathbb{C}^3 \) to \( T \). The coordinate axes are exactly the irreducible subrepresentations and they are non-isomorphic (each is one copy of a different character). It follows that every \( w \in N_G(T) \) must permute these subspaces and every \( t \in Z_G(T) \) must act on each subspace separately. But these subspaces are irreducible, so each \( t \in Z_G(T) \) must be diagonal, and hence an element of \( T \). It follows that \( T = Z_G(T) \) so it is a maximal torus, that \( N_G(T) \) is the group of signed permutations, and that \( W(G : T) = N_G(T)/T \simeq S_3 \).

Differentiating the definition \( G = \{ g \in \text{SL}_3(\mathbb{C}) \mid g^* g = \text{Id} \} \) we see that \( \mathfrak{g} = \{ X \in \mathbb{C} \mid X^* + X = i \} \) that is the set of anti-Hermitian matrices of trace zero. Since every \( Y \in \mathbb{C} \) can be uniquely written in the form

\[
Y = \frac{Y + Y^\dagger}{2} + \frac{Y - Y^\dagger}{2} = \frac{Y - Y^\dagger}{2i} + \frac{Y + Y^\dagger}{2} \in \mathfrak{g} \oplus i\mathfrak{g}
\]

we see that \( \mathfrak{g}_\mathbb{C} \simeq \mathbb{C}^3 \). It is also clear that \( t = \{ i \text{diag}(\theta_1, \theta_2, \theta_3) \mid \theta_1 + \theta_2 + \theta_3 = 0 \} \).

Now for \( i \neq j \) let \( E_{ij} \in \mathbb{C} \subset \mathfrak{m}_{3\mathbb{C}}(\mathbb{C}) \) be the matrix with zeroes everywhere except that \( (E_{ij})_{ij} = 1 \). Then for \( H = i \text{diag}(\theta_1, \theta_2, \theta_3) \) we have \( \text{ad}_H \cdot E_{ij} = i(\theta_i - \theta_j)E_{ij} \) so the roots of \( G \) are the maps \( e_{ij}(H) = \theta_i - \theta_j \).

To find the Weyl chamber we note that the Frobenius, or Hilbert–Schmidt norm on \( M_3(\mathbb{C}) \) is \( \text{U}(3) \)-invariant. In terms of this norm (and removing the factor of \( i \)) an orthonormal basis of \( t \) is given by \( \frac{1}{\sqrt{6}} \text{diag}(1,1,-2), \frac{1}{\sqrt{2}} \text{diag}(1,-1,0) \). Now for

\[
H = \frac{x}{\sqrt{6}} \text{diag}(1,1,-2) + \frac{y}{\sqrt{2}} \text{diag}(1,-1,0)
\]

26
we have
\[ e_{12}(H) = \sqrt{2}y \]
\[ e_{23}(H) = \frac{\sqrt{3}}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y \]
\[ e_{13}(H) = \frac{\sqrt{3}}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y. \]

In the coordinates \((x, y)\) we therefore have:
\[ u_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, u_{23} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}, u_{13} = \begin{pmatrix} \sqrt{3}/2 \\ +1/2 \end{pmatrix}. \]

These three lines are the lines at slopes \(\frac{\pi}{3}\) and \(\frac{2\pi}{3}\) through the origin, dividing \(\mathbb{R}^2\) into six identical sectors. We call these sectors \(\text{Weyl chambers}\), the lines \(\text{walls}\), and note that \(S_3\) (which has order 6) acts on the six chambers simply transitively.

**Exercise 135.** Do the same for \(SU(n), SO(2n), SO(2n + 1), Sp(n)\).

### 3.5.5. Weyl chambers.

The complement of hyperplane \(u_\alpha\) consists of two half-spaces: the sets \(\{H \in t \mid \alpha(H) > 0\}\) and \(\{H \in t \mid \alpha(H) < 0\}\). It follows that the connected components of \(t \setminus \bigcup_{\alpha \in \Phi} u_\alpha\) are intersections of half-spaces, hence convex cones.

**Definition 136.** These connected components are called the (open) \(\text{Weyl chambers}\) in \(t\). We call \(u_\alpha\) a \(\text{wall}\) of the chamber \(C\) if \(\dim(u_\alpha \cap \bar{C}) = \text{rk} G - 1\). More generally, a (codimension-\(k\)-) \(\text{facet}\) of the Weyl chamber \(C\) is any non-empty set of the form \(F = (u_{\alpha_1} \cap \cdots \cap u_{\alpha_k} \cap \bar{C})^\circ\) where the interior is taken as a subset of the vector space \(u_{\alpha_1} \cap \cdots \cap u_{\alpha_k}\). We note that the closure \(\bar{C}\) is the disjoint union of the facets of \(C\) (where \(C\) itself is the unique facet of codimension zero).

**Remark 137.** Note that we are studying the Weyl chambers in \(\text{liet}\), rather than the Weyl chambers in \(t^*\) where the root system lies.

Given a chamber \(C\), let \(\Delta\) be the set of roots \(\alpha\) such that \(u_\alpha\) is a wall of \(C\) and such that \(\alpha\) is positive on \(C\) (note that \(u_\alpha = u_{-\alpha}\) and that exactly one of \(\alpha, -\alpha\) is positive on \(C\)).

**Fact 138.** The chamber is exactly the set bounded by the walls: \(C = \{H \in t \mid \forall \alpha \in \Delta : \alpha(H) > 0\}\).

**Observation 139.**

1. The Weyl group acts on \(G\) by automorphisms while fixing \(T\). It therefore permutes the roots, hence their kernels, and hence the Weyl chambers.
2. It is clear that there is a bijection between Weyl chambers and (satisfiable) notions of positivity (choices of sign for all the \(\alpha(H)\)). The facets are determined by having some roots positive, some negative, and some vanishing.

**Lemma 140.** The group \(W' = \langle \{s_\alpha\}_{\alpha \in \Delta} \rangle\) acts transitively on the set of Weyl chambers.
PROOF. Fix \( x \in C \); let \( C' \) be any other chamber and let \( y \in C' \). Note that (being equivalence classes for an equivalence relation) if two chambers intersect they are equal, to it suffices to show that \( wy \in C \) for some \( w \in W' \). For this choose \( w \) such that \( \|wy - x\| \) is minimal. If \( wy \not\in C \) then by Fact 138 above, there is a wall \( u_\alpha \) such that \( x, wy \) are on opposite sides of \( \alpha \). Decomposing \( x, wy \) into their components along and perpendicular to \( u_\alpha \) it is then clear that
\[
\|s_\alpha(wy) - x\| < \|wy - x\|
\]
which is a contradiction since \( s_\alpha w \in W' \).

\[\square\]

**Lemma 141.** The group \( W \) acts simply transitively on the chambers.

**Proof.** We already know the action is transitive. Suppose \( w \in N_G(T) \) stabilizes the chamber \( C \). Since \( W \) is finite, \( w \) has finite order as an automorphism of \( T \) so averaging over a \( w \)-orbit shows that \( w \) fixes some \( x \in C \) (recall that \( C \) is convex). Think of \( x \) as an element \( H \in \text{liet} \), we have that \( \text{Ad}_w H = H \), that is \( w \in Z_G(H) \).

On the other hand, since \( H \in C \), \( \alpha(H) \neq 0 \) for all \( \alpha \in \Phi \). It follows that \( H \) acts non-trivially in every root space so \( ZG_C(H) = t_C \) and hence \( ZG(H) = \text{liet} \). Now \( ZG(H) \) is connected (this is true for all \( H \in \text{liet} \)); its lie Algebra being \( ZG(H) \) we conclude that \( ZG(H) = T \) and hence \( w \in T \). It follows that the image of \( w \) in \( W = N_G(T)/T \) is trivial.

\[\square\]

**Corollary 142.** \( W' = W \), that is the algebraic and analytic Weyl groups coincide.

**Proof.** Let \( w \in W \). By the transitivity of \( W' \) there is \( w' \in W' \) such that \( w \cdot C = w' \cdot C \). By the simplicity of the action we conclude \( w = w' \in W' \).

**\[\square\]**

**3.5.6. Geometry of the roots.** The linear map \( s_\alpha - \text{Id}_t \) is non-zero but vanishes on \( u_\alpha \). It therefore has rank 1, and factors through \( \alpha \). We conclude that there is a unique \( \tilde{\alpha} \in t \) such that
\[
s_\alpha(x) = x - \alpha(x) \tilde{\alpha}.
\]
The dual action on \( t^* \) is then
\[
s_\alpha(v) = v - v(\tilde{\alpha}) \alpha
\]
and since \( s_\alpha(\alpha) = -\alpha \) we have \( \alpha(\tilde{\alpha}) = 2 \).

**Definition 143.** Call \( \tilde{\alpha} \) the coroot associated to \( \alpha \) and write \( \tilde{\Phi} \) for the set of coroots.

**Remark 144.** If \( \alpha + \beta \) is a root it need not be the case that \( \alpha + \beta = \tilde{\alpha} + \tilde{\beta} \). In particular, a root system and its dual need not be isomorphic.

**Exercise 145.** \( (t/3, \tilde{\Phi}) \) is a root system, the dual root system.

**Lemma 146.** Coroots are integral, that is \( \tilde{\alpha} \in \Lambda = \text{Ker}(\exp \uparrow t) \).

**Proof.** The element \( \frac{1}{2} \tilde{\alpha} \) has exp \( (2\pi i \alpha(\frac{1}{2} \tilde{\alpha})) = \exp(2\pi i) = 1 \) since \( \alpha(\tilde{\alpha}) = 2 \). In other words, \( \exp \left( \frac{1}{2} \tilde{\alpha} \right) \) lies in the kernel the exponential root \( \chi_\alpha \) and hence is \( \text{Ad}(s_\alpha) \)-stable. On the other hand, \( s_\alpha(\tilde{\alpha}) = -\tilde{\alpha} \) so \( \text{Ad}(s_\alpha) \exp \left( \frac{1}{2} \tilde{\alpha} \right) = \exp \left( -\frac{1}{2} \tilde{\alpha} \right) \). It follows that
\[
\exp \left( \frac{1}{2} \tilde{\alpha} \right) = \exp \left( -\frac{1}{2} \tilde{\alpha} \right),
\]
that is \( \exp(\tilde{\alpha}) = 1 \) and \( \tilde{\alpha} \in \Lambda \).

\[\square\]

**Corollary 147.** For any \( \alpha, \beta \in \Phi \) we have \( n_{\alpha \beta} = \beta(\tilde{\alpha}) \in \mathbb{Z} \).
**Definition 148.** The $n_{\alpha\beta}$ are called the Cartan numbers of $\mathfrak{g}$. Note that $s_\alpha(\beta) = \beta - n_{\alpha\beta}\alpha$.

**Definition 149.** The coroot lattice is the subgroup $\Gamma < \Lambda$ generated by the coroots.

**Fact 150.** $\Lambda/\Gamma \simeq \pi_1(G)$.

**Corollary 151.** $\check{G}$ is compact iff $\pi_1(G)$ is finite iff $\check{\Phi}$ spans $\mathfrak{t}$ iff $z = 0$ iff $Z(G)$ is finite. In each of those equivalent cases we say that $G$ is semisimple.

**Fact 152.** $G$ is semisimple iff its lie algebra is the direct sum of nonabelian simple lie algebras, iff $G$ is the almost direct product of nonabelian quasisimple groups.

Recall that we have equipped $\mathfrak{g}$ with an invariant inner product. This also endows $\mathfrak{t}^*$ with an inner product and then

$$s_\alpha(v) = v - 2\frac{\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

so if we identify $\text{liet}$, $\mathfrak{t}^*$ using this inner product the element $\check{\alpha}$ is identified with $\check{\alpha}^* = \frac{2\alpha}{\langle \alpha, \alpha \rangle} \in \mathfrak{t}^*$. Now $n_{\alpha\beta} = \beta(\check{\alpha}) = \langle \beta, \check{\alpha}^* \rangle = 2\frac{\beta}{\langle \alpha, \alpha \rangle}$. It follows that

$$n_{\alpha\beta}n_{\beta\alpha} = 4\frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} \leq 4$$

by Cauchy–Schwartz, with equality iff $\alpha, \beta$ are proportional. Since the two Cartan numbers are integers, each is zero iff $\alpha \perp \beta$, and if both are non-zero their product is positive, we see that (up to exchanging $\alpha, \beta$) if $\alpha, \beta$ are not proportional, the pair $(n_{\alpha\beta}, n_{\beta\alpha})$ must be one of the seven possibilities:

$$(0, 0), \pm (1, 1), \pm (1, 2), \pm (1, 3).$$

In each case the pair $(n_{\alpha\beta}, n_{\beta\alpha})$ determines the angle between the roots and (if they are not orthogonal) the ratio of their lengths.

**Corollary 153.** Let $\alpha, \beta$ be non-proportional and suppose that that $n_{\alpha\beta} > 0$ (equivalently that $\langle \alpha, \beta \rangle > 0$). Then $\alpha - \beta \in \Phi$.

**Proof.** If $n_{\alpha\beta} > 0$ then either $n_{\beta\alpha} = 1$, at which point $s_\beta(\alpha) = \alpha - \beta \in \Phi$, or $n_{\alpha\beta} = 1$, at which point $s_\beta(\alpha) = \beta - \alpha \in \Phi$ and then $\alpha - \beta \in \Phi$ as well. $\square$

### 3.5.7. Simple roots

Fix a Weyl chamber $C$, giving a notion of positivity: call $\alpha \in \Phi$ positive if it is positive on $C$, negative otherwise, and write $\Phi^+, \Phi^-$ for the sets of positive and negative roots. Since roots have constant sign on $C$ it suffices to evaluate them at a fixed $H \in C$.

**Definition 154.** Call $\alpha \in \Phi^+$ simple if it is not a sum of positive roots, and let $\Delta$ be the set of simple roots.

**Remark 155.** This clearly depends on the choice of $C$. More on that anon.

**Lemma 156.** Every positive root is a positive sum of simple roots.

**Proof.** Let $\alpha$ be a counterexample with $\alpha(H)$ minimal. Then $\alpha$ is not a simple root, so $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Phi^+$. But then $\beta(H) + \gamma(H) = \alpha(H)$ shows that $\beta(H), \gamma(H) < \alpha(H)$ so they are sums of positive roots and we have a contradiction. $\square$

**Proposition 157.** $\Delta \subseteq \mathfrak{t}^*$ is linearly independent.
PROOF. Let \( \alpha, \beta \in \Delta \) be distinct. If the angle between them was acute \( (\langle \alpha, \beta \rangle > 0) \) then by Corollary 153 one of \( \alpha - \beta, \beta - \alpha \) would be a positive root and this would make either \( \alpha \) or \( \beta \) decomposable. It follows that \( \langle \alpha, \beta \rangle \leq 0 \) for each pair. They are also all contained in the half-plane \( \{ v \mid v(H) > 0 \} \). We show these two hypotheses suffice to make a set of vectors independent. Indeed, suppose we have a linear dependence in \( \Delta \). We then have disjoint non-empty \( A, B \subset \Delta \) and positive coefficients \( \{ a_\alpha \}_{\alpha \in A}, \{ b_\beta \}_{\beta \in B} \) such that
\[
\sum_{\alpha \in A} a_\alpha \cdot \alpha = \sum_{\beta \in B} b_\beta \cdot \beta .
\]
Call this vector \( v \). Then
\[
0 \leq \langle v, v \rangle = \sum_{\alpha, \beta} a_\alpha b_\beta \langle \alpha, \beta \rangle \leq 0
\]
and it follows that \( v = 0 \). We therefore have
\[
0 = v(H) = \sum_{\alpha \in A} a_\alpha \cdot \alpha(H) > 0 ,
\]
a contradiction. □

LEMMA 158. \( \Delta \) spans \( (g/\mathfrak{z})^* \).

PROOF. Every simple root vanishes on \( \mathfrak{z} \), so the same holds for every element of the span. Conversely, the span contains \( \Phi \); it follows the common kernel of the span is exactly \( \mathfrak{z} \) so the span is exactly \( (g/\mathfrak{z})^* \). □

COROLLARY 159. \#\( \Delta \) is the semisimple rank.

LEMMA 160. \( \{ u_\alpha \}_{\alpha \in \Delta} \) are the walls of \( C \).

PROOF. \( \{ H \mid \forall \alpha \in \Delta : \alpha(H) > 0 \} = \{ H \mid \forall \alpha \in \Phi^+ : \alpha(H) > 0 \} = C \). Since \( \Delta \) are independent they are all walls. □

DEFINITION 161. A system of simple roots (or simple system) is a subset \( \Delta \subset \Phi \) such that every root is either the sum of elements of \( \Delta \) or the negative of such a sum.

COROLLARY 162. Every system of simple roots is the set of walls of a Weyl chamber, we have a bijection between systems of simple roots, notions of positivity, and Weyl chambers, and the Weyl group acts transitively on simple systems. In particular, every root belongs to a simple system.
CHAPTER 4
Semisimple Lie groups
CHAPTER 5

Representation theory of real groups
Functional Analysis

In this appendix we review the basics of topological vector spaces. References include TVS.

A.1. Topological vector spaces

Let $K$ be a non-discrete complete valued field

**Definition 163.** A **topological vector space** is a vector space $V$ over $K$ equipped with a topology so that $(V,+)$ is a topological group and such that scalar multiplication is a continuous map $\cdot : K \times V \to V$.

**Proposition 164.** A finite-dimensional $K$-vector space has a unique topology making it into a TVS. In particular, if $V,W$ are TVS with $V$ finite-dimensional then $\text{Hom}_K(V,W) = \text{Hom}_{cts}(V,W)$ and if $V \subset W$ then $V$ is closed and complete. If $K$ is locally compact then a TVS over $K$ is locally compact iff it is finite-dimensional.

**Definition 165.** Fix a TVS $V$. Call $C \subset V$:

1. **Balanced**, if $\alpha v \in C$ for all $x \in C$, $|\alpha| \leq 1$.
2. **Absorbing**, if $\bigcup_{t>0} tC = V$ (that is, for all $v \in V$ there are $u \in C$ and $t > 0$ such that $tu = v$).
3. **Bounded**, if for every open neighbourhood $W \ni 0$ there is $t > 0$ such that $C \subset tW$.
4. **Totally bounded**, if for every open neighbourhood $W \ni 0$ there is a finite set $\{u_i\}_{i=1}^n \subset V$ such that $C \subset \bigcup_i (u_i + W)$.

**Lemma 166.** Every finite subset of a TVS is bounded.

**Lemma 167.** Every TVS has a basis neighbourhoods of 0 which are balanced.

**Definition 168.** A net $\{x_\alpha\}_{\alpha \in D} \subset V$ is called a **Cauchy net** if for every neighbourhood $W$ of 0 there is $\delta \in D$ such that if $\alpha, \beta \geq \delta$ then $x_\alpha - x_\beta \in W$. $X \subset V$ is **complete** if every Cauchy net in $X$ converges to a limit in $X$. $V$ is **quasi-complete** if every closed bounded subset of $X$ is complete.

**Lemma 169.** In a quasi-complete TVS every totally bounded subset is relatively compact.

**Assumption 170.** $K = \mathbb{R}$ or $\mathbb{C}$.

**Definition 171.** Fix a TVS $V$. Call $C \subset V$ **convex**, if $tu + (1-t)v \in C$ for all $u,v \in C$, $t \in [0,1]$. Call $V$ locally convex if any neighbourhood of 0 contains a convex neighbourhood of zero.

**Proposition 172.** A TVS is locally convex iff its topology is determined by a family of seminorms.

**Lemma 173.** Let $V$ be locally convex, $C \subset V$ be totally bounded. Then the convex hull and balanced convex hull of $C$ are also totally bounded.

**Corollary 174.** Let $V$ be locally convex and quasi-complete and let $C \subset V$ be compact. Then the closed convex hull of $C$ is compact.
DEFINITION 175. The continuous dual of $V$ is $V' \overset{\text{def}}{=} \text{Hom}_{\text{cts}}(V, K)$.

THEOREM 176 (Hahn–Banach). Let $V$ be locally convex, $U \subset E$ a subspace, $f \in U'$. Then $f$ has a continuous linear extension to $V$. In particular, $V'$ separates the points of $V$.

A.2. Quasicomplete locally convex TVS

[based on Casseleman, Garrett]

PROPOSITION 177. An inverse limit of quasi-complete spaces is quasi-complete. The direct product of a family of quasi-complete space is quasi-complete. The weak-* dual of a Banach space is quasi-complete.

Let $V$ be a locally convex TVS.

DEFINITION 178. Let $\Omega$ be a measureable space.

(1) Call $f : \Omega \to V$ weakly measurable if $\varphi \circ f : \Omega \to K$ is measurable for each $\varphi \in V'$. Let $\mu$ be a measure on $\Omega$ and let $f : \Omega \to V$ be weakly measurable. Call $v \in V$ the Gelfand–Pettis integral of $f$ (and write $v = \int f \, d\mu$) if for every $\varphi \in V'$ $\varphi \circ f$ is $\mu$-integrable and we have

$$\varphi(v) = \int_\Omega \varphi \circ f \, d\mu.$$  

REMARK 179. Note that the integral clearly exists as an element of $V''$; the question is about existence as an element of $V$. Since $V'$ separates the points, it is also clear that the integral (if it exists) is unique.

THEOREM 180. Let $V$ be quasi-complete, let $\Omega$ be compact, $\mu$ a Radon measure, and let $f : \Omega \to V$ be continuous. Then $\int f \, d\mu$ exists.

PROOF. Wlog $\mu$ is a probability measure. In that case we also show $\int f \, d\mu$ lies in the closed convex hull of $f(\Omega)$.

LEMMA 181. If $V$ is finite-dimensional then $\int f \, d\mu$ exists and lies in the convex hull of $f(\Omega)$.

Write $C$ for the closed convex hull of $f(\Omega)$. For every finite $\mathcal{F} \subset V'$ consider the continuous linear map $\mathcal{F} : V \to K^\mathcal{F}$ given by $v \mapsto (\varphi(v))_{\varphi \in \mathcal{F}}$. It maps $C$ continuously onto the convex hull of the image of $\mathcal{F} \circ f$. Now $\int_\Omega (\mathcal{F} \circ f) \, d\mu$ exists in that convex hull, and we obtain a non-empty closed convex subset

$$C_\mathcal{F} = \left\{ v \in C \mid \mathcal{F}(v) = \int_\Omega (\mathcal{F} \circ f) \, d\mu \right\}.$$  

Since $\bigcap_{i=1}^r C_{\mathcal{F}_i} = C_{\bigcup_{i=1}^r \mathcal{F}_i}$ we see that this family has the finite intersection property, and it follows that

$$\bigcap_{\mathcal{F}} C_\mathcal{F}$$  

is non-empty. The (necessarily unique) point there is the desired integral. \qed

A.3. Integration

A.4. Spectral theory and compact operators

A.5. Trace-class operators and the simple trace formula