1. Finding the expansions

(1) \( f(x) = x^3 + 3x + 1 \), to third order. \( f'(x) = 3x^2 + 3 \), \( f''(x) = 6x \), \( f'''(x) = 6 \), all further derivatives are zero.

(a) Expand about \( x = 1 \): \( f(1) = 5 \), \( f'(1) = 6 \), \( f''(1) = 6 \), \( f'''(1) = 6 \). Get (actual equality since \( f \) is a polynomial)

\[
f(x) = 5 + 6(x - 1) + \frac{6}{2!} (x - 1)^2 + \frac{6}{3!} (x - 1)^3.
\]

(b) Expand about \( x = 5 \): \( f(5) = 141 \), \( f'(5) = 78 \), \( f''(5) = 30 \), \( f'''(5) = 6 \). Get (actual equality since \( f \) is a polynomial)

\[
f(x) = 141 + 78(x - 1) + \frac{30}{2!} (x - 1)^2 + \frac{6}{3!} (x - 1)^3.
\]

(2) Let’s try \( \sin(11x + x^2) \) to third order. We know \( \sin(u) \approx u - \frac{u^3}{3!} \) to third order. Now \( 11x + x^2 \) vanishes at zero so we can plug in and get:

\[
\sin(11x + x^2) \approx (11x + x^2) - \frac{(11x + x^2)^3}{3!}
\]

\[
= 11x + x^2 - \frac{1}{6} (11^3 x^3 + 3(11x)^2 x^2 + 3(11x)(x^2)^2 + (x^2)^3)
\]

\[
\approx 11x + x^2 - \frac{1331}{6} x^3
\]

to third order (the \( x^4 \), \( x^5 \), \( x^6 \) terms are negligible when working to third order).

(3) Let’s try \( \sin(11x + 5) \) to third order. (aside: \( 11^2 = 121, 11^3 = 1331 \)).

(a) About \( x = -\frac{5}{11} \), this reads \( \sin \left( 11 \left( x - \left( -\frac{5}{11} \right) \right) \right) \) we plug in: \( 11(x - a) - \frac{(11(x-a))^3}{3!} = 11 \left( x + \frac{5}{11} \right) + \frac{1331}{6} \left( x + \frac{5}{11} \right)^3 \).

(b) About \( x = 0 \), using derivatives. The first three are \( 11 \cos(11x + 5), -11^2 \sin(11x+5), -11^3 \cos(11x+5) \) at \( 0 \) we get \( \sin(5), 11 \cos(5), -121 \sin(5), -1331 \cos(5) \) so to third order about \( x = 0 \),

\[
\sin(11x + 5) \approx \sin(5) + 11 \cos(5) \cdot x - \frac{1121 \sin(5)}{2} x^2 - \frac{1331 \cos(5)}{6} x^3.
\]

(c) About \( x = 0 \), using addition formula and substitution. Recall \( \sin((11x + 5) = \sin(5) \cos(11x) + \cos(5) \sin(11x) \). To third order, \( \cos(u) = 1 - \frac{u^2}{2} \),

\[
\sin(u) = u - \frac{u^3}{3}
\]

so

\[
\sin(11x + 5) \approx \sin(5) \left[ 1 - \frac{(11x)^2}{2} \right] + \cos(5) \left[ (11x) - \frac{(11x)^3}{3!} \right]
\]

\[
= \sin(5) + 11 \cos(5) \cdot x - \frac{121 \sin(5)}{2} x^2 - \frac{1331 \cos(5)}{6} x^3
\]
after rearranging.

(4) \( E(v) = \frac{m c^2}{\sqrt{1-v^2/c^2}} \)
the expression for the energy of a relativistic particle of mass \( m \) and velocity \( v \). Let’s expand to second order, to see what happens at velocities much smaller than the speed of light \( c \).

(a) By the chain rule, \( E'(v) = mc^2 \left( \left( \frac{1}{2} \right) \left( 1 - \frac{v^2}{c^2} \right)^{-3/2} \right) \left( -\frac{2v}{c^2} \right) \)
so \( E'(0) = 0 \). Next, by the quotient rule
\[
E''(v) = \frac{1 \cdot \left( 1 - \frac{v^2}{c^2} \right)^{3/2} + v^2/2 \cdot \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} \cdot \left( -\frac{2v}{c^2} \right)}{(1 - \frac{v^2}{c^2})^3}
\]
\[
= \frac{m}{(1 - \frac{v^2}{c^2})^{3/2}} - 3m \frac{v^2/c^2}{(1 - \frac{v^2}{c^2})^{5/2}}
\]

We get \( E''(0) = m \). The second-order expansion is therefore
\[
E(v) \approx mc^2 + \frac{1}{2} mv^2,
\]
recovering the classical kinetic energy at low velocities.

(b) Different approach: let \( u = \frac{v^2}{c^2} \). Get \( E(u) = mc^2 (1 - u)^{-1/2} \). Again \( E(0) = mc^2 \), also
\[
\frac{dE}{du} = mc^2 \frac{1}{2} (1 - u)^{-3/2}
\]
\[
\frac{d^2E}{du^2} = mc^2 \frac{3}{2} \frac{1}{2} (1 - u)^{-5/2}
\]
\[
\frac{d^3E}{du^3} = mc^2 \frac{3 \cdot 5}{2 \cdot 2} (1 - u)^{-7/2}
\]
and so on. Get:
\[
E(u) = mc^2 \left[ 1 + \frac{1}{2} u + \frac{1}{2!} \frac{1 \cdot 3}{2 \cdot 2} u^2 + \frac{1}{3!} \frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2} u^3 + \cdots \right]
\]
so plugging in \( u = \frac{v^2}{c^2} \), get
\[
E(v) = mc^2 \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \left( \frac{v^2}{c^2} \right)^2 + \frac{5}{16} \left( \frac{v^2}{c^2} \right)^3 + \cdots \right]
\]
\[
= mc^2 + \frac{1}{2} mv^2 + mc^2 \left[ \frac{3}{8} \left( \frac{v^2}{c^2} \right)^2 + \frac{5}{16} \left( \frac{v^2}{c^2} \right)^3 + \cdots \right].
\]

Remark: It is very useful to keep the rest of the series in terms of the small parameter \( \frac{v^2}{c^2} \) instead of in terms of \( v^2 \). We get the series of relativistic corrections to the classical Newtonian formula \( \frac{1}{2} mv^2 \).

(5) Example: Suppose we know \( f'(x) = f(x) \) and \( f(0) = 1 \). What is the Taylor expansion?

Solution: If \( f'(x) = f(x) \) then \( f''(x) = f'(x) = f(x) \) and \( f^{(k+1)}(x) = \frac{d}{dx} f^{(k)}(x) = \frac{d}{dx} f(x) = f(x) \) by induction. So \( f^{(k)}(0) = 1 \) for all \( k \). So
\[
f(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\]
Remark: this looks silly: we know that $f'(x) = e^x$. But the same approach applies to $f'(x) = f(x) + f^2(x)$. Then $f'(0) = 2$, and

$$f''(x) = f'(x) + 2f(x)f'(x) = f(x) + f^2(x) + 2f(x) \left( 2(f(x) + f^2(x)) \right) = f(x) + 3f^2(x) + 2f^3(x)$$

so $f''(0) = 6$ and we get to second order $f(x) \approx 1 + 2x + 3x^2$ with no formula for $f$. 