MATH 100: MORE EXAMPLES OF TAYLOR EXPANSIONS

1. Finding the expansions

(1) $f(x) = x^3+3x+1$, to third order. $f'(x) = 3x^2+3$, $f''(x) = 6x$, $f'''(x) = 6$, all further derivatives are zero.

(a) Expand about $x = 1$: $f(1) = 5$, $f'(1) = 6$, $f''(1) = 6$,$f'''(1) = 6$. Get (actual equality since f is a polynomial)

$$\begin{aligned} f(x) &= 5 + 6(x-1) + \frac{6}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 \\ &= 5 + 6(x-1) + 3(x-1)^2 + (x-1)^3 . \end{aligned}$$

(b) Expand about $x = 5$: $f(5) = 141$, $f'(1) = 78$, $f''(1) = 30$,$f'''(1) = 6$. Get (actual equality since f is a polynomial)

$$\begin{aligned} f(x) &= 141 + 78(x-1) + \frac{30}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 \\ &= 141 + 78(x-1) + 15(x-1)^2 + (x-1)^3 . \end{aligned}$$

(2) Let's try $\sin(11x + x^2)$ to third order. We know $\sin(u) \approx u - \frac{u^3}{3!}$ to third order. Now $11x + x^2$ vanishes at zero so we can plug in and get:

$$\begin{aligned} \sin(11x + x^2) &\approx (11x + x^2) - \frac{(11x + x^2)^3}{3!} \\ &= 11x + x^2 - \frac{1}{6}\left(11^3x^3 + 3(11x)^2x^2 + 3(11x)(x^2)^2 + (x^2)^3\right) \\ &\approx 11x + x^2 - \frac{1331}{6}x^3 \end{aligned}$$

to third order (the x^4, x^5, x^6 terms are negligible when working to third order).

(3) Let's try $\sin(11x + 5)$ to third order. (aside: $11^2 = 121$, $11^3 = 1331$).

(a) About $x = -\frac{5}{11}$ this reads $\sin\left(11\left(x - (-\frac{5}{11})\right)\right)$ we plug in: $11(x-a) - \frac{(11(x-a))^3}{3!} = 11\left(x + \frac{5}{11}\right) + \frac{1331}{6}\left(x + \frac{5}{11}\right)^3$.

(b) About $x = 0$, using derivatives. The first three are $11\cos(11x + 5), -11^2\sin(11x+5), -11^3\cos(11x+5)$ at 0 we get $\sin(5), 11\cos(5), -121\sin(5), -1331\cos(5)$ so to third order about $x = 0$,

$$\sin(11x + 5) \approx \sin(5) + 11\cos(5)\cdot x - \frac{121\sin(5)}{2}x^2 - \frac{1331\cos(5)}{6}x^3 .$$

(c) About $x = 0$, using addition formula and substitution. Recall $\sin(11x + 5) = \sin(5)\cos(11x) + \cos(5)\sin(11x)$. To third order, $\cos(u) = 1 - \frac{u^2}{2}$, $\sin(u) = u - \frac{u^3}{3}$ so

$$\begin{aligned} \sin(11x + 5) &\approx \sin(5)\left[1 - \frac{(11x)^2}{2}\right] + \cos(5)\left[(11x) - \frac{(11x)^3}{3!}\right] \\ &= \sin(5) + 11\cos(5)\cdot x - \frac{121\sin(5)}{2}x^2 - \frac{1331\cos(5)}{6}x^3 \end{aligned}$$

after rearranging.

(4) $E(v) = \frac{mc^2}{\sqrt{1-v^2/c^2}}$, the expression for the energy of a relativistic particle of mass m and velocity v. Let's expand to second order, to see what happens at velocities much smaller than the speed of light c.

(a) By the chain rule, $E'(v) = mc^2\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right) = mv\left(1-\frac{v^2}{c^2}\right)^{-3/2}$ so $E'(0) = 0$. Next, by the quotient rule

$$\begin{aligned} E''(v) &= m\frac{1\cdot\left(1-\frac{v^2}{c^2}\right)^{3/2} + v\frac{3}{2}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(-\frac{2v}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)^3} \\ &= m\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} - 3m\frac{v^2/c^2}{\left(1-\frac{v^2}{c^2}\right)^{5/2}}\,. \end{aligned}$$

We get $E''(0) = m$. The second-order expansion is therefore

$$E(v) \approx mc^2 + \frac{1}{2}mv^2\,,$$

recovering the classical kinetic energy at low velocities.

(b) Different approach: let $u = \frac{v^2}{c^2}$. Get $E(u) = mc^2(1-u)^{-1/2}$. Again $E(0) = mc^2$, also

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}u} &= mc^2\frac{1}{2}(1-u)^{-3/2} \\ \frac{\mathrm{d}^2E}{\mathrm{d}u^2} &= mc^2\frac{1}{2}\frac{3}{2}(1-u)^{-5/2} \\ \frac{\mathrm{d}^3E}{\mathrm{d}u^3} &= mc^2\frac{1}{2}\frac{3}{2}\frac{5}{2}(1-u)^{-7/2} \end{aligned}$$

and so on. Get:

$$E(u) = mc^2\left[1 + \frac{1}{2}u + \frac{1}{2!}\cdot\frac{1\cdot 3}{2\cdot 2}u^2 + \frac{1}{3!}\frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}u^3 + \cdots\right]$$

so plugging in $u = \frac{v^2}{c^2}$, get

$$\begin{aligned} E(v) &= mc^2\left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \frac{5}{16}\left(\frac{v^2}{c^2}\right)^3 + \cdots\right] \\ &= mc^2 + \frac{1}{2}mv^2 + mc^2\left[\frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \frac{5}{16}\left(\frac{v^2}{c^2}\right)^3 + \cdots\right]. \end{aligned}$$

Remark: It is very useful to keep the rest of the series in terms of the *small parameter* $\frac{v^2}{c^2}$ instead of in terms of v^2. We get the series of relativistic corrections to the classical Newtonian formula $\frac{1}{2}mv^2$.

(5) Example: suppose we know $f'(x) = f(x)$ and $f(0) = 1$. What is the Taylor expansion?

Solution: If $f'(x) = f(x)$ then $f''(x) = f'(x) = f(x)$ and $f^{(k+1)}(x) = \frac{d}{dx}f^{(k)}(x) = \frac{d}{dx}f(x) = f(x)$ by induction. So $f^{(k)}(0) = 1$ for all k. So

$$f(x) \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Remark: this looks silly: we know that $f'(x) = e^x$. But the same approach applies to $f'(x) = f(x) + f^2(x)$. Then $f'(0) = 2$, and

$$f''(x) = f'(x)+2f(x)f'(x) = f(x)+f^2(x)+2f(x)\left(2(f(x)+f^2(x)\right) = f(x)+3f^2(x)+2f^3(x)$$

so $f''(0) = 6$ and we get to second order $f(x) \approx 1 + 2x + 3x^2$ with no formula for f.