MATH 100: MORE EXAMPLES FOR NEWTON'S LAW OF COOLING

(1) (Final 2011) A wealthy man was found murdered in his home at 10pm at night. The temperature of his body was $33°C$, and of the room $21°C$. An hour later the temperature of the body was $31°C$. Assume the body cools after death according to Newton's Law of Cooling.

(a) Normal body temperature is $37°C$. When did the murder take place?
Solution: Let $T(t)$ be the temperature of the body in degrees Celsius, t hours after 10pm. Let $y(t) = T(t) - 21$ be the temperature difference from the room. We then have $y(t) = Ce^{-kt}$ for some C, k. We are given that $C = y(0) = 33 - 21 = 12$ and $Ce^{-k} = y(1) = 31 - 21 = 10$ so $12e^{-k} = 10$, that is $\frac{12}{e^k} = 10$ which means $e^k = \frac{12}{10}$. Taking logarithms we get $k = \log \frac{12}{10} = \log(1.2)$. We need to find t such that $y(t) = 37 - 21 = 16$. This was t such that $12e^{-kt} = 16$ so $e^{-kt} = \frac{16}{12}$. Taking logarithms we find

$$-kt = \log \frac{16}{12}$$

and using $k = \log(1.2)$ that

$$-t_{\text{death}} = \frac{\log\left(1\frac{1}{3}\right)}{\log\left(1\frac{1}{5}\right)}.$$

In short, the murder took $\frac{\log(4/3)}{\log(6/5)}$ hours before 10pm.

(b) Did the murder occur before 9pm? Justify your answer.
Straightforward solution: the logarithm is a monotone function, so $0 = \log 1 < \log\left(1\frac{1}{5}\right) < \log\left(1\frac{1}{3}\right)$. It follows that

$$\frac{\log\left(1\frac{1}{3}\right)}{\log\left(1\frac{1}{5}\right)} > 1$$

so the death occured *more than an hour* before 10pm, that is before 9pm.

Over-complicated solution: We need to approximate $\frac{\log\left(1+\frac{1}{3}\right)}{\log\left(1+\frac{1}{5}\right)}$. For this let $f(x) = \log(1 + x)$. Then $f(0) = \log 1 = 0$ and $f'(x) = \frac{1}{1+x}$ so $f'(0) = 1$ and the linear approximation to f at 0 is $f(x) = x$. We therefore guess $\log\left(1 + \frac{1}{3}\right) \approx \frac{1}{3}$, $\log\left(1 + \frac{1}{5}\right) \approx \frac{1}{5}$ so the death happened about $\frac{1/3}{1/5} = \frac{5}{3}$ hours before 10pm, which would have been *before* 9pm. To justify this we show that our approximation is not too big. The second derivative $f''(x) = -\frac{1}{(1+x)^2}$ is negative, so the error in the linear approximation (which is $\frac{f''(c)}{2}x^2$for some c) is negative, that is$\log\left(1+\frac{1}{5}\right) \leq \frac{1}{5}$, which means

$$\frac{1}{\log\left(1+\frac{1}{5}\right)} \geq 5\,.$$

We also note that $f''(0) = -1$ so the quadratic approximation to log is $\log(1+x) \approx x - \frac{1}{2}x^2$ and now the error term involves $f'''(x) = \frac{2}{(1+x)^3} > 0$. It follows that the quadratic approximation is an under-estimate: that

$$\log\left(1+\frac{1}{3}\right) \geq \frac{1}{3} - \frac{1}{2}\left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{1}{18} = \frac{5}{18}\,.$$

Multiplying the two estimates gives:

$$-t_{\text{death}} = \frac{\log\left(1\frac{1}{3}\right)}{\log\left(1\frac{1}{5}\right)} \geq \frac{5/18}{1/5} = \frac{25}{18} > 1\,.$$