FOURIER SERIES AND THE POISSON SUMMATION FORMULA  
(NOTES FOR MATH 613)  
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NOTATION

Write $S^1$ for the group $\{z \in \mathbb{C}^\times \mid |z| = 1\}$. For $z \in \mathbb{C}$ write $e(z) \overset{\text{def}}{=} e^{2\pi iz}$. All group homomorphisms are assumed to be continuous.

For a topological space $X$ write $C(X)$ for the space of $\mathbb{C}$-valued continuous functions on $X$, $C_c(X)$ for the subspace of functions of compact support. If $\mu$ is a Radon measure on $X$ and $1 \leq p \leq \infty$ write $L^p(\mu)$ for the usual space of $p$-integrable functions. We sometimes write $L^p(X)$ when the measure is clear (and note that if $L^p(f \mu) = L^p(\mu)$ if $f$ is bounded).

When $X$ is compact, $C(X)$ is complete in the $L^\infty$ norm and (Stone-Weierstrass) a subalgebra $\mathcal{A} \subset C(X)$ is dense if it separates points, does not have a common zero, and is closed under conjugation.

On a manifold $X$ write $C^j(X)$ for the space functions differentiable $j$ times with continuous derivatives of order $j$, $C^\infty(X) = \cap_j C^j(X)$, and $C^\infty_c(X) = C^\infty(X) \cap C_c(X)$.

On $\mathbb{R}^n$ say $f$ is of rapid decay if $f(x) (1 + \|x\|)^N$ is bounded for all $N$. Say $f \in C^\infty(\mathbb{R}^n)$ is of Schwartz class if $f$ and all its derivatives are of rapid decay.

1. FOURIER SERIES AND FOURIER INVERSION ON $\mathbb{R}^n/\Lambda$

Let $V$ be an inner product space, fix a lattice $\Lambda < V$, and write $\mathbb{T}$ for the torus $V/\Lambda$. Let $\Lambda^*$ be the dual lattice.

**Definition 1.** $L^2(\mathbb{T})$ and $L^2(\Lambda^*)$ will denote the spaces with respect to the Haar probability measure and counting measure, respectively.

**Problem 2.** (Functional analysis)

1. Show that $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.
2. Show that $C_c(\Lambda^*)$ is dense in $L^2(\Lambda^*)$.

**Problem 3.** (Trigonometric polynomials)

1. Show that $k \mapsto (x \mapsto e(kx))$ is an injective group homomorphism $\Lambda^* \hookrightarrow \text{Hom}(\mathbb{T}, S^1)$.
2. Show that the characters $e(kx)$ are linearly independent in $C(\mathbb{T})$.
   
   Hint: Evaluate a linear combination $\sum a_k e(kx) = 0$ of shortest length at two different values of $x$.
3. Let $\mathcal{P}$ be the algebra of continuous functions on $\mathbb{T}$ generated by the $e(kx)$. Show that $\mathcal{P}$ is simply the linear span of these characters.
4. Let $x \in \mathbb{T}$ be non-zero. Show that there exists $k \in \Lambda^*$ such that $e(kx) \neq 1$.
   
   Hint: $\Lambda^{**} = \Lambda$. 

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Problem 8. (Orthogonality of characters)

(1) For \( k \in \Lambda^* \) show that 
\[
\frac{1}{\text{vol}(\mathbb{T})} \int_{\mathbb{T}} e(kx) \, dx = \begin{cases} 
1 & k = 0 \\
0 & k \neq 0.
\end{cases}
\]

(2) Conclude that for \( k, l \in \Lambda^* \) one has 
\[
\frac{1}{\text{vol}(\mathbb{T})} \int_{\mathbb{T}} e(kx) e(lx) \, dx = \delta_{kl}.
\]

Definition 5. For \( g \in C_c(\Lambda^*) \) set \( \hat{g}(x) = \sum_{k \in \Lambda^*} e(kx) \).

Problem 6. (The inverse map) We show that \( g \mapsto \hat{g} \) extends to an isometric isomorphism \( L^2(\Lambda^*) \to L^2(\mathbb{T}) \).

(1) (Parseval’s identity) Show that \( \|\hat{g}\|_{L^2(\mathbb{T})} = \|g\|_{L^2(\Lambda^*)} \), that is that 
\[
\frac{1}{\text{vol}(\mathbb{T})} \int_{\mathbb{T}} |\hat{g}(x)|^2 \, dx = \sum_{k \in \Lambda^*} |g(k)|^2.
\]

(2) Since \( C_c(\Lambda^*) \) is dense in \( L^2(\Lambda^*) \), conclude that \( g \mapsto \hat{g} \) extends to an isometric embedding \( L^2(\Lambda^*) \to L^2(\mathbb{T}) \), and show that the image is a closed subspace.

(3) Let \( f \in L^2(\mathbb{T}) \) be of norm one and orthogonal to the image of this map. Approximating \( f \) by a trigonometric polynomial show that \( \langle f, f \rangle = 0 \) and derive a contradiction.

Definition 7. For \( f \in L^2(\mathbb{T}) \) and \( k \in \Lambda^* \) set \( \hat{f}(k) = \frac{1}{\text{vol}(\mathbb{T})} \int_{\mathbb{T}} f(x) e(-kx) \, dx \).

Problem 8. (The direct map)

(1) Show that \( |\hat{f}(k)| \leq \|f\|_{L^2(\mathbb{T})} \). Conclude that \( |\hat{f}(k)| \leq \|f\|_{L^\infty(\mathbb{T})} \) also.

(2) For \( g \in C_c(\Lambda^*) \) show that \( \hat{g}(k) = g(k) \). Show that the same holds for \( g \in L^2(\Lambda^*) \).

(3) Conclude that the map \( f \mapsto \hat{f} \) takes values in \( L^2(\Lambda^*) \) and is the inverse to the map \( g \mapsto \hat{g} \).

Problem 9. (Smooth functions)

(1) Integrating by parts, show that for \( k \neq 0 \) and \( f \in C^2(\mathbb{T}) \) we have 
\[
|\hat{f}(k)| \leq \frac{1}{|2\pi k|} \| \triangle f \|_{L^\infty(\mathbb{T})}.
\]

(2) Assume now that \( f \in C^\infty(\mathbb{T}) \). Show that \( F^{(\alpha)}(x) = \sum_{k \in \Lambda^*} (2\pi i k)^\alpha \hat{f}(k) e(kx) \) converges uniformly for all multi-indices \( \alpha \).

(3) Integrating term-by-term show that \( F^{(\alpha)} \) is the \( \alpha \)th derivative of \( F^{(0)} \).

(4) Show that \( F^{(0)} = f \) pointwise.

2. The Poisson Summation Formula

Definition 10. For \( f \in L^1(\mathbb{V}) \) and \( k \in \mathbb{V}^* \) set \( \hat{f}(k) = \int_{\mathbb{V}} f(x) e(-kx) \, dx \) and call this the Fourier transform of \( f \).

Problem 11. (The Fourier transform) Let \( f \in L^1(\mathbb{R}^n) \)

(1) Show that \( \|\hat{f}\|_{L^\infty(\mathbb{V}^*)} \leq \|f\|_{L^1(\mathbb{V})} \).

(2) Show that \( \hat{f} \in C(\mathbb{V}) \).

*Hint:* The bounded convergence theorem.

(3) On \( V = \mathbb{R} \) let \( f = \exp(-|x|) \). Show that \( \hat{f}(k) = \frac{2}{1 + 4\pi^2 k^2} \).
(4) Let \( \Re(\alpha) > 0 \) and let \( f(x) = \exp \{ -\pi \alpha x^2 \} \). Show that \( \hat{f}(k) = \sqrt{\frac{1}{\alpha}} \exp \{ -\frac{\pi}{\alpha} k^2 \} \) where we take the branch of the square root with a cut at \((-\infty, 0]\).

\[ \text{Hint: Shift contours to reduce the problem to the known formula} \int_{\mathbb{R}} \exp \left( -\alpha x^2 \right) dx = \sqrt{\frac{\pi}{\alpha}}. \]

(5) Let \( Q \in M_n(\mathbb{R}) \) be a positive-definite symmetric matrix, and let \( f(x) = \exp \left( -2\pi \langle x | Q | x \rangle \right) \).

Show that \( \hat{f}(k) = 2^{-n/2} (\det Q)^{-1/2} \exp \left\{ -2\pi \langle k | Q^{-1} | k \rangle \right\} \).

Finally, let \( f \in C^\infty(\mathbb{R}^n) \) and its derivatives decay polynomially, quickly enough that \( \Pi_A f \) converges absolutely to a smooth function.

**Problem 12.** (The Poisson Summation Formula) Let \( f \in L^1(\mathbb{R}^n) \) decay quickly enough that \( \Pi_A f \in \mathbb{C}(V/\Lambda) \).

(1) For \( k \in \Lambda^* \) show that \( \hat{\Pi_A f}(k) = \frac{1}{\text{vol}(\Lambda)} \hat{f}(k) \) where the first hat is the Fourier transform on \( \mathbb{T} \) and the second is the one on \( V \).

(2) Show that \( \Pi_A f(x) = \frac{1}{\text{vol}(\Lambda)} \sum_{k \in \Lambda^*} \hat{f}(k) e(kx) \). Conclude that:

\[ \sum_{v \in \Lambda} f(v) = \frac{1}{\text{vol}(\Lambda)} \sum_{k \in \Lambda^*} f(k). \]

3. **The Fourier transform and Fourier inversion on \( \mathbb{R}^n \)**

**Problem 13.** (The Schwartz class) Let \( f \in S(V) \), \( \Lambda < V \) a fixed lattice.

(1) Differentiating under the integral sign show that \( \hat{f}(k) \) is smooth.

(2) Integrating by parts show that then \( \hat{f} \) is of rapid decay.

(3) Combining the two calculations show that \( \hat{f} \in S(V) \).

(4) Applying the PSF to \( f \) with the lattice \( r\Lambda \) and taking \( r \to \infty \) show that

\[ f(0) = \int_{V^*} \hat{f}(k) dk. \]

(5) Let \( g(x) = f(x+y) \). Show that \( \hat{g}(k) = \hat{f}(k)e(ky) \) and conclude that

\[ f(x) = \int_{V^*} \hat{f}(k)e(kx) dk. \]

(6) Use the same methods to establish *Parseval’s identity*

\[ \|f\|_{L^2(V)} = \|\hat{f}\|_{L^2(V^*)}. \]

(7) Conclude that the Fourier transform extends to a bijective isometry \( \mathcal{F} : L^2(V) \to L^2(V^*) \).