Quantum Unique Ergodicity

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FIG. 2. Left column, three scarred states of the stadium; right column, the isolated, unstable periodic orbits corresponding to the scars.
Other examples

(Images: Bäcker, Stromberg)
Quantum Unique Ergodicity

**Problem:** What happens as $\lambda \to \infty$? What is a “feature”?

**Pointwise** How big does $\|u_\lambda\|_\infty$ get as $\lambda \to \infty$?

**Weakly** What happens to $\int |u_\lambda|^2 f \, d\text{vol}$ as $\lambda \to \infty$?

**Theorem (Schnirel’man–Zelditch–Colin de Verdière)**

*If the billiard dynamics is chaotic (ergodic) then for almost all eigenfunctions* $\int |u_\lambda|^2 f \, d\text{vol} \to \frac{1}{\text{vol}} \int f \, d\text{vol}$

**Conjecture (Rudnick–Sarnak)**

*On a manifold of negative sectional curvature, replace “almost all” with “all”.*

Hassell 2008: For stadium billiard, can’t remove “almost”.
Plan

1. Bounds on eigenfunctions on the tree and in the plane
2. “Classical” and “quantum” mechanics
3. “Arithmetic” QUE
4. Without arithmetic
5. Negative results for approximate eigenfunctions
A pointwise bound

**Theorem (Hörmander bound)**

\[ \| u_\lambda \|_\infty \leq C \lambda^{\frac{n-1}{4}} \| u_\lambda \|_2. \]

**Proof (in spirit).**

Use \( u_\lambda \) as the initial condition for an evolution equation, e.g.

\[ i \hbar \frac{\partial}{\partial t} \psi(t, x) = -\Delta_x \psi(t, x). \]

- \( \psi(t, x) = e^{-i\lambda t} u_\lambda(x) \) is a solution.
- But solutions tend to follow classical trajectories.
- So \( \psi(t, x) \) looks like \( u_\lambda \) “averaged” over a region near \( x \), and can relate \( \psi(t, x) \) to \( \| u_\lambda \|_2 \).
Some physics
The Space of Lattices

Move to \textit{curved geometry} and \textit{periodic boundary conditions}.

- $\mathcal{P}_n = \{\text{symmetric, positive-definite } n\text{-matrices } X, \det(X) = 1\}$
- $\text{SL}_n(\mathbb{R})$ acts by $g \cdot X := gXg^t$, preserving metric:
  \[\text{dist}(\text{Id}, X) = \left(\sum_{i=1}^{n} |\log \mu_i|^2\right)^{1/2}, \mu_i = \text{eigenvalues}.\]
- For $n = 2$, $\mathcal{P}_n$ is the hyperbolic plane.
- Study the quotient $\mathcal{L}_n = \text{SL}_n(\mathbb{Z})\backslash \mathcal{P}_n$
  $= \text{isometry classes of unimodular lattices in } \mathbb{R}^n$. 
Arithmetic QUE

- Domain has *number-theoretic symmetries*, manifest as *Hecke operators* \( T_p f = \sum_{y \sim x} f(y) \)

\[
T_p \Delta = \Delta T_p, \quad T_p T_q = T_q T_p
\]

- Study limits of joint eigenfunctions. Start with \( n = 2 \):
  - Rudnick–Sarnak 1994: limits don’t scar on closed geodesics.
  - Iwaniec–Sarnak 1995: savings on Hörmander bound
    - small balls have small mass
  - Bourgain–Lindenstrauss 2003: limits have positive entropy
    - small dynamical balls have small mass
  - Lindenstrauss 2006: from this get equidistribution.
Higher-rank QUE

- What about $n \geq 3$?
- No longer negatively curved – extend Rudnick–Sarnak conjecture
- S–Venkatesh 2007: limits respect \textit{Weyl chamber flow}
- S–Venkatesh: (non-degenerate) limits are uniformly distributed if $n$ is prime (division algebra quotient).

QUE Results proceed by
- Lift to the bundle where classical flow lives.
- Bound mass of dynamical balls ("positive entropy")
- Apply measure-classification results to identify the limit.
QUE on general manifolds

- In $\mu(B(C,\epsilon)) \ll \epsilon^h$, $h$ measures the complexity of $\mu$.
- Related to the metric entropy $h(\mu)$.
- Anantharaman ~2003: On a manifold of negative curvature, every quantum limit has positive entropy.
- Anantharaman + others: quantitative improvements
- Idea: “quantum partition”
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Introduction
1 PlanarExercise
2 Classical and quantum mechanics
3 Arithmetic eigenfunctions
4 Without arithmetic
5 Scarring for quasimodes

Applied to the space of lattices

- \( \mathcal{L}_n \) not negatively curved (has flats).
- Nevertheless limits have positive entropy:
  - Microlocal calculus adapted to locally symmetric spaces.
  - Entropy contribution from “rapidly expanding” directions.
- Measure-classification
  - Restriction on possible ergodic components.
  - Use *quantitative* entropy bound.

**Theorem (Anantharaman–S)**

Let \( X = \Gamma \backslash \mathcal{P}_3 \) be compact. Then every quantum limit on \( X \) is at least \( \frac{1}{4} \) Haar measure.
New uncertainty principle

- Density is now known for $n = 2$:

**Theorem (Dyatlov–Jin 2018)**

Every quantum limit on a compact hyperbolic surface has full support.

**Theorem (Dyatlov–Jin–Nonnenmacher 2019)**

The same on a compact surface with Anosov geodesic flow.
Approximate eigenfunctions

- Method of Anantharaman applies to approximate eigenfunctions.

\[
\| \Delta u_\lambda + \lambda u_\lambda \| \leq C \frac{\sqrt{\lambda}}{\log \lambda}
\]

- Entropy depends on \( C \).

**Problem**

*What are the possible limits of these “log-scale quasimodes”?

\[
\| \Delta u_\lambda + \lambda u_\lambda \| \leq C \frac{\sqrt{\lambda}}{\log \lambda}
\]
Scarring of quasimodes

Problem

On a manifold $M$, construct log-scale quasimodes which concentrate on singular measures

\[
\|\Delta u_\lambda + \lambda u_\lambda\| \leq C \frac{\sqrt{\lambda}}{\log \lambda}
\]

\[
\lim_{\lambda \to \infty} \int |u_\lambda|^2 f \, d\text{vol} = \int f \, d\mu
\]

- Brooks 2015: $M =$ hyperbolic surface, $\mu =$ geodesic.
  - Uses the geometry explicitely (Eisenstein packets)
- Eswarathasan–Nonnenmacher 2016: $M =$any surface, $\mu =$ hyperbolic geodesic.
High dimensions

Theorem (Eswarathasan–S 2017)

Let $M$ be a hyperbolic manifold, and let $N \subset M$ be a compact totally geodesic submanifold. Then there is a sequence of log-scale quasimodes uniformly concentrating on $N$.

- Includes the case $N = \text{closed geodesic}$.
- Actually, any quantum limit on $N$ achievable.

Corollary

$(M \text{ compact})$ every invariant measure on $M$ is a limit of log-scale quasimodes.

Proof.

In a hyperbolic system, closed orbits are dense in the space of invariant measures.