Solutions to Sample Midterm 1 for Math 104 and Math 184

[10] 1.
Compute the following limits:

a) (2 marks) $\lim_{x\to 3} \frac{x^2-9}{x^2-8x+15}$

$$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(x-5)}$$

$$= \frac{3+3}{3-5}$$

$$= \frac{6}{-2}$$

$$= 3$$

b) (3 marks) $\lim_{x\to 1} \frac{x-1}{\sqrt{2x-1}-1}$

$$= \lim_{x\to 1} \frac{(x-1)(\sqrt{2x-1}+1)}{\underbrace{(\sqrt{2x-1}-1)(\sqrt{2x-1}+1)}_{(2x-1)-1}}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(\sqrt{2x-1}+1)}{2\cancel{(x-1)}}$$

$$= \frac{\sqrt{2\cdot 1-1}+1}{2}$$

$$= 1$$

Compute the derivatives of the following functions.

c) (2 marks) Find $f'(x)$ where $f(x) = (x^2 + 7x)(e^x + x^3 + 2x^2 + 1)$.
DO NOT SIMPLIFY YOUR ANSWER.

$$f'(x) = (2x+7)(e^x + x^3 + 2x^2 + 1) + (x^2 + 7x)(e^x + 3x^2 + 4x)$$

d) (3 marks) Find $h'(1)$ where $h(x) = \dfrac{xf(x) - 5}{g(x)}$, $f'(1) = 2$, $g'(1) = -3$ and $f(1) = 1$, $g(1) = 1$. EXPRESS YOUR ANSWER AS AN INTEGER.

$$h'(x) = \frac{(f(x) + xf'(x))g(x) - (xf(x) - 5)g'(x)}{(g(x))^2}$$

$$\therefore h'(1) = \frac{(1 + 1\cdot 2)\cdot 1 - (1\cdot 1 - 5)(-3)}{1^2}$$

$$= -9$$

[7] **2.** Let $f(x)$ be a function defined for all x near some number a.

(a) (2 marks) Carefully state the definition of $f'(a)$, the derivative of $f(x)$ at the point $x = a$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

OR

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) (5 marks) Suppose $f(x) = \frac{2}{x-1}$. Show that $f'(a) = \frac{-2}{(a-1)^2}$ using the definition of the derivative. NO credit will be given for any other method.

$$f'(a) = \lim_{x \to a} \frac{\frac{2}{x-1} - \frac{2}{a-1}}{x - a}$$

$$= \lim_{x \to a} \frac{2(a-1) - 2(x-1)}{(x-1)(a-1)(x-a)}$$

$$= \lim_{x \to a} \frac{-2\cancel{(x-a)}}{(x-1)(a-1)\cancel{(x-a)}}$$

$$= \frac{-2}{(a-1)(a-1)}$$

$$= \frac{-2}{(a-1)^2}$$

[8] 3. A spaceship travels along a path given by the graph $y = x^2 \sin(x)$ in the plane.

a) (2 marks) What is the y coordinate of the ship when $x = \pi$?

b) (6 marks) At the point in part (a), a piece breaks off of the ship and travels along the tangent line to the ship's path at this point. What is the y coordinate of this piece when its x coordinate is $x = 2\pi$?

a) when $x = \pi$, $y = \underbrace{\pi^2 \sin \pi} = 0$

b) $$\frac{dy}{dx} = 2x \sin x + x^2 \cos x$$

at $x = \pi$,

$$\left.\frac{dy}{dx}\right|_{\pi} = 2\pi \underbrace{\sin \pi}_{0} + \pi^2 \underbrace{\cos \pi}_{-1} = -\pi^2$$

$\therefore$ The equation of the tangent line is

$$y - 0 = -\pi^2 (x - \pi)$$

i.e. $y = -\pi^2 (x - \pi)$

Then the y-coordinate of the piece at $x = 2\pi$ is

$$y = -\pi^2 (2\pi - \pi) = -\pi^3$$

[8] 4. ABC Inc. has recently introduced the ABC smartphone. They anticipate that if they sell the smartphone at the price of \$300 per unit, they will sell 5000 units per week. For each \$10 increase in the price, they anticipate selling 200 fewer units per week. The fixed costs of producing the smartphone are \$100000 per week, and each smartphone costs ABC \$50 to make.

a) (2 marks) Let p be price and q be weekly demand for the smartphone. Find the linear demand function $p(q)$.

$$\frac{p-300}{q-5000} = -\frac{1}{20}$$

$$\therefore\ p - 300 = -\frac{1}{20}(q - 5000)$$

$$p = -\frac{1}{20}(q-5000) + 300 = -\frac{1}{20}q + 550$$

b) (1 marks) Find the weekly cost function $C(q)$.

$$C(q) = 100\ 000 + 50q$$

c) (2 marks) The weekly profit function $P(q)$ is given by $P(q) = -\frac{1}{20}q^2 + 500q - 100000$. Find the marginal weekly profit function $MP(q)$ (The Marginal Profit function is just the derivative of the profit function: $MP(q) = P'(q)$).

$$MP(q) = -\frac{1}{10}q + 500$$

d) Suppose that the price is currently \$200 per unit. If the price is increased by a small amount,

i) (1 marks) Will the quantity demanded increase or decrease? (explain)

ii) (2 marks) Will the weekly profit increase or decrease? (explain)

i) decrease since slope of p,q curve is negative (from part a))

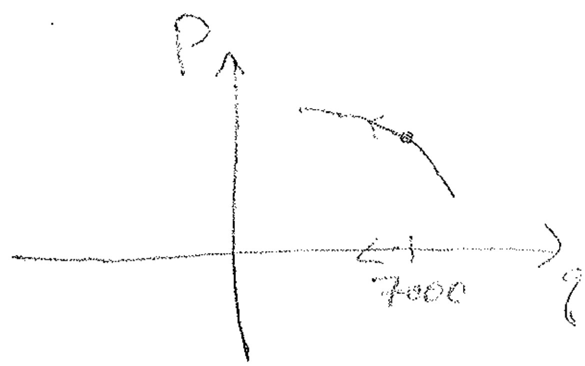
ii) $p = 200 \Rightarrow q = 7000$ using part a)

$MP(7000) = -200 < 0$ using part c)

$\therefore$ as p increases (from 200)

$\Rightarrow$ q decreases (from part i)

$\Rightarrow$ P increases



[8] 5. Consider the function

$$f(x) = \begin{cases} ax^2 + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ bx^3 + cx & \text{if } x > 1. \end{cases}$$

(a) (2 marks) Compute the left and right hand limits of $f(x)$ as $x \to 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} ax^2+1 = a+1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} bx^3+cx = b+c$$

(b) (2 marks) What equations must a, b and c satisfy so that f is continuous at $x = 1$?

Need $a+1 = b+c = f(1) = 2$

$\Rightarrow$ $a = 1$, $b+c = 2$

(c) (4 marks) What equations must a, b and c satisfy for f to be differentiable at $x = 1$? Determine the values of a, b, c for which f will be differentiable at $x = 1$.

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 2ax = 2a$$

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} 3bx^2+c = 3b+c$$

$\Rightarrow$ $2a = 3b+c$, $a = 1$ and $b+c = 2$

$\therefore$ $a = 1$, $b = 0$, $c = 2$

[9] 6. For each question below, either explain why the statement is true or show the statement is false by providing a counter example if appropriate. No credit will be given for answers without justification.

(a) The equation $\frac{x^3 - 2\sqrt{x} - 5}{e^x} = 0$ has a solution. TRUE

$f(x)$

Since

$f(x)$ is continuous on $(-\infty, \infty)$

$f(4) = \frac{64 - 4 - 5}{e^4} > 0$, $f(0) = \frac{-5}{e^0} < 0$

$\therefore f(c) = 0$ for some c in $(0, 4)$ by I.V.T.

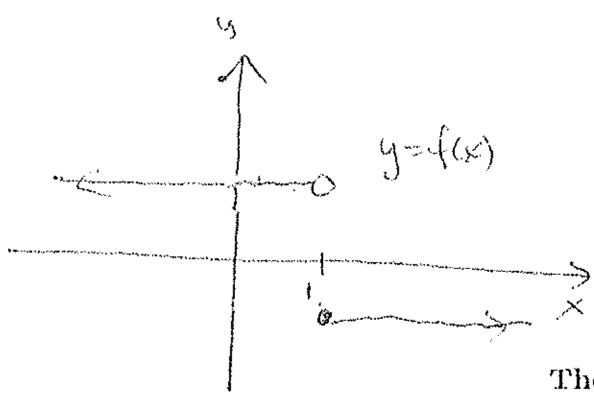
(b) The following function is continuous at $x = 3$ TRUE

$$f(x) = \begin{cases} \frac{x^2+1}{x-1}, & \text{if } x \neq 3, \\ 5, & \text{if } x = 3, \end{cases}$$

Since

$\lim_{x \to 3} f(x) = \frac{3^2+1}{3-1} = \frac{10}{2} = 5 = f(3)$

(c) If $f(x) + g(x)$ is differentiable at $x = 1$, then both $f(x)$ and $g(x)$ must also be differentiable at $x = 1$. FALSE



let $g(x) = -f(x)$

Then $f(x) + g(x) = 0$ for all x

$\therefore f(x) + g(x)$ diff. at $x = 1$.

But neither $f(x)$, $g(x)$ diff at $x=1$.

The End